Network Formation and Systemic Risk*

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Abstract

This paper introduces a model of endogenous network formation and systemic risk. In it, firms form joint ventures called 'links' which are subsequently subjected to either good or bad shocks. Bad shocks incentivize default. Links yield full benefits only if the counter-party does not subsequently default on the project. Accordingly, defaults triggered by bad shocks render firms insolvent and defaults propagate via links. The model yields three insights. First, stable networks with ex-ante identical agents exhibit a core-periphery structure. Second, an increase in the probability of good shocks increases systemic risk. Third, because the network formed depends on the correlation between shocks to links, an observer who misconceives the correlation will underestimate the probability of system-wide default by a factor of a half.

JEL classification: D85, G01.

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1 Introduction

The awkward chain of events that so upset the bankers in 2008 began with the collapse of Lehman Brothers. Panic spread, the dollar wavered, and world markets fell. Interconnectedness of the financial system, it was suggested, allowed the Lehman Brothers' fall to threaten the stability of the entire system. The possibility that the failure of one firm will trigger the widespread failure of otherwise healthy firms is called systemic risk. Even the failure of a non-financial firm, such as General Motors, will, via its suppliers, have spillovers into the non-automotive sector. Indeed, the 2009–2014 restructuring plan filed by General Motors says:

> "The systemic risk to the automotive industry and the overall U.S. economy are considerable, just as the bankruptcy of Lehman had a ripple effect throughout the financial industry ..."

These events inspired scholars to characterize network structures conducive to systemic risk, which is the risk that shocks to a part of the system propagate and damage the entire system. With some exceptions, these papers assume an exogenously given network. A node (or subset of them) is subjected to a shock and the propagation of the shock across the network is studied. Absent are reasons for the presence of links between nodes. In this paper, a link between two nodes represents a potentially lucrative joint project. However, each link increases the possibility of contagion. In the presence of a trade-off between being exposed to systemic risk and having more projects, we ask what kinds of networks would be formed? Systemic risk is not unique to financial networks. It is a concern in supply chains and the web of firms linked by joint projects or trade credits. In these examples, other externalities are present, but they are *not* a focus of this paper.

We propose a *simple* model of contagion that spreads across agents whom we call firms. In the model, firms form links and become counterparties. Links represent joint ventures, and each link is subjected to a good or a bad shock. We assume the networks formed are stable in the sense that no subset of firms has an incentive to deviate and choose a different set of links. A formal definition appears in Section 2.2.

A bad shock to a link makes firms incident to that link default. Moreover, any firm that has a defaulting counter-party also defaults. We also allow direct shocks to firms (called node shocks), which can be interpreted as idiosyncratic risk of default. Systemic risk is the probability of the event that every firm chooses to default. Section 4 describes a detailed micro foundation of this model in terms of firms borrowing from outside lenders and investing as pairs into projects.

The main insights are enumerated below.

1. Core-periphery networks emerge despite systemic risk.

Core-periphery networks are considered ubiquitous; see, for example, Bech and Atalay (2010) and Craig and Von Peter (2014). They consist of a subset of nodes (the core) that are densely connected to each other and a periphery of nodes partitioned into clusters that are all connected to the core but 'lightly' connected to each other. Examples of such networks are displayed in Figure 1. Prior work has shown how core-periphery networks can emerge endogenously systemic risk (except Farboodi (2015) and Erol (2016)). Such structures may facilitate systemic risk because the core, with its dense interconnections, encourage contagion. Financial institutions that occupy the core, for example, are thought to be contributors to systemic risk for this very reason.[1]

In this paper, stable networks exhibit a core-periphery structure *even* when firms are ex-ante identical. The core arises because of a coordination failure between lenders and firms that leads to heterogeneous interest rates for firms, as described in Section 4. Firms better able to withstand counter-party failures and link shocks occupy the core acting as a barrier to contagion between distinct components in the periphery. However, each firm, including those in the core, is susceptible to failure caused by idiosyncratic node shocks. Accordingly, the core makes the default risk of the entire system highly correlated and becomes a major source of systemic risk.

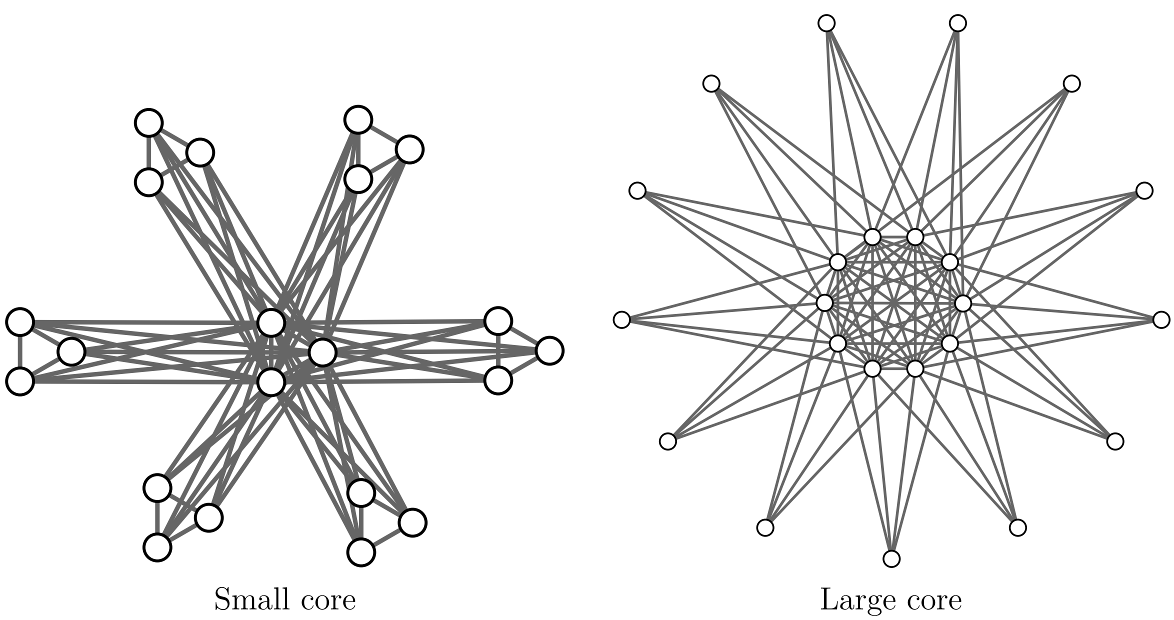


Figure 1: Structure of the stable network: core-periphery

[1]High level of interconnectedness of certain institutions is seen as a systemic threat. See, for example, http://www.fsb.org/what-we-do/policy-development/systematically-important-financial-institutions-sifis/.

2. Volatility Paradox

If the probability of a good shock to a link increases, the network becomes more interconnected at a rate at which systemic risk always increases. This is a network version of the volatility paradox of Brunnermeier and Sannikov (2014): low volatility leads investors to behave in ways that make the financial system more fragile and prone to crisis. We show that the volatility paradox persists, even when we relax the condition that a single defaulting counterparty causes its own default and network externalities are mild. In the absence of contagion, there is no volatility paradox, therefore, it emerges because of contagion.

3. Systemic risk vs. systematic risk: What can we learn from the network about the probability of system-wide failure?

Systematic risk is the risk of system-wide failure due to common exposures of institutions to risks outside the system, whereas systemic risk is the risk that the system fails via the propagation of bad shocks. When shocks are perfectly correlated, i.e., under systematic risk, the network formed is strongly interconnected. When shocks are idiosyncratic, i.e., under systemic risk, such a strongly interconnected network is formed only when shocks are likely to be good. Accordingly, an endogenously formed and strongly interconnected network implies an upper bound on the underlying systemic risk, whereas such a network is uninformative about the underlying systematic risk. One who observes a very dense network must understand the correlation structure in order to correctly assess the probability of system-wide failure.

We think this relevant to the debate between two theories of financial destruction advanced to explain the 2008 financial crisis. The first, described above, is dubbed the 'domino theory' of systemic risk. The alternative, advocated by Edward Lazear,[2] is dubbed the 'popcorn' theory of systematic risk. Lazear describes it thusly in a 2011 opinion piece in the *Wall Street Journal*:

> "The popcorn theory emphasizes a different mechanism. When popcorn is made (the old-fashioned way), oil and corn kernels are placed in the bottom of a pan, heat is applied and the kernels pop. Were the first kernel to pop removed from the pan, there would be no noticeable difference. The other kernels would pop anyway because of the heat. The fundamental structural cause is the heat, not the fact that one kernel popped, triggering others to follow.
>
> Many who believe that bailouts will solve Europe's problems cite the Sept. 15, 2008 bankruptcy of Lehman Brothers as evidence of what allowing one domino

[2]Chair of the U.S. President's Council of Economic Advisers during the 2007-2008 financial crisis.

to fall can do to an economy. This is a misreading of the historical record. Our financial crisis was mostly a popcorn phenomenon. At the risk of sounding defensive (I was in the government at the time), I believe that Lehman's downfall was more a result of the factors that weakened our economic structure than the cause of the crisis."

Related literature

This paper contributes to four streams in the economic analysis of networks. We summarize them here. Detailed comparisons to prior work can be found in the body of the paper.

Systemic Risk in Networks: Much prior work, such as Acemoglu et al. (2015), Eboli (2019), Elliott et al. (2014), Gai et al. (2011), and Glasserman and Young (2015), takes the network as exogenous. We consider fully strategic network formation. Acemoglu et al. (2015) contains a discussion of network formation but within a set of limited alternatives. Babus (2016) also has a model of network formation, but one in which agents share the goal of minimizing the probability of system-wide default. In our model, agents are concerned with their expected payoffs and only indirectly with the possibility of system wide failure.

Our network formation model is closest to Blume et al. (2013). However, in that paper, the risk of a node initially defaulting to start contagion is independent of the network formed because the exogenous shocks hit nodes. In our model, the likelihood of a node initially defaulting to start contagion depends on the network, in particular, the degree of the node, because the exogenous shocks hit links.

Contagion in Networks: Our paper is also a contribution to the literature on contagion in networks (see, for example, Morris (2000) and Goyal and Vega-Redondo (2005)). It extends this literature by endogenizing the networks and incorporating uncertain payoffs.

Core-Periphery Structure: Many financial networks exhibit a core-periphery structure. A variety of explanations have been offered. Farboodi et al. (2017), for example, shows that a core-periphery structure emerges in a search model where some agents choose to trade faster than others. Wang (2016) demonstrates that a core-periphery structure emerges in an inter-dealer market with ex-ante identical agents as the result of a trade-off between trade competition and inventory efficiency. None of these papers account for contagion and systemic risk. Two papers that do are Farboodi (2015) and Erol (2016). The agents in these papers are ex-ante heterogenous. This paper generates a core-periphery structure with ex-ante identical agents.

Structure

For ease of exposition, we begin with a reduced form of the model in which payoffs are given exogenously. Section 2 considers the simplest setting of ex-ante identical firms and link shocks only. We characterize group stable networks, discuss how systemic risk varies with the probability of a good shock and the volatility paradox. This setting, as we show, corresponds to the periphery in our larger model.

In Section 3 we enlarge the model to include firms with varying (and exogenously given) degrees of resilience to contagion and node shocks. Under group stability, firms that are more resilient form the core of the network, the less resilient occupy the periphery.

Section 4 micro-founds the model in Section 3 in terms of firms borrowing from outside lenders and investing as pairs into projects. It also endogenizes the degree of resilience of each firm to shocks. In this way, we show how a core-periphery structure can emerge even with ex-ante identical agents. Section 5 concludes.

2 The Periphery

2.1 The Model of the Periphery

Denote by $N = \{1, 2, ..., n\}$ a finite set of *firms*. In what follows we will sometimes refer to firms as nodes. A *link* $\{i, j\}$ represents a joint venture between firms i and j. $E \subset [N]^2$ is the set of links and (N, E) is the *network*. If $\{i, j\} \in E$, then i and j are called *counterparties*. Denote by $D_i = \{j \in N | \{i, j\} \in E\}$ the set of counterparties of i and $d_i = |D_i|$ the *degree* of i. There exists $\tilde{d} \leq n - 1$ such that $d_i \leq \tilde{d}$ for all i. For each link $\{i, j\} \in E$, two outcomes s_{ij} and s_{ji} are chosen by nature identically and independently. s_{ij} represents the net benefit of firm i in its partnership with j. With probability $\alpha \in (0, 1)$, the link is *good for* i: $s_{ij} = \theta$ where $\theta > 0$ is a constant. With probability $1 - \alpha$, the link is *bad for* i: $s_{ij} = \theta - \gamma$ where $\gamma > 0$ is a constant. Denote by b_i the number of links that are bad for i.

Upon the realization of shocks, each firm i either *continues* or *defaults*. If i continues, it receives the sum of $\sum_{j \in D_i} s_{ij}$. Furthermore, for each $j \in D_i$, if j defaults, i's benefit from its link with j is reduced by $\beta > 0$, from s_{ij} to $s_{ij} - \beta$. This captures the extra costs that must be incurred to cover for j's role in the project. Denote the number of defaulting counterparties of i by f_i . i's payoff from continuation is $u_i = d_i\theta - f_i\beta - b_i\gamma$. If u_i is non-negative, i continues. If u_i is negative, i defaults and gets an outside option 0. Losses due to bad shocks (γ's) can trigger defaults, which can propagate via counter-party losses (β's). This describes the contagion, which is exogenous for now.

Assumption 1. $\tilde{d}\theta < min\{\beta, \gamma\}$.

To see the implication of Assumption 1 recall that $d_i \leq \tilde{d}$ for all $i \in N$. Therefore, under $\tilde{d}\theta < \gamma$, one bad shock to an incident link causes the firm to drop all its projects. Because $\tilde{d}\theta < \beta$, a single defaulting counter-party causes a firm to drop all its projects. In short, a single bad shock to any link triggers a contagion, and no firm is able to withstand either a shock or a defaulting counterpart.

All firms that are connected face a risk of contagion, and this is the only risk they face.[3] The links provide no diversification benefit.[4] The consequences of relaxing Assumption 1 are discussed in Appendix B.2.

To summarize, each firm enjoys a benefit from each direct link to other firms, but also bears a default risk from each direct and indirect linkage to other firms. Whenever, one of the links in the component containing the firm is hit by a shock, then all firms in the component default. The next proposition states that the expected payoff of a firm depends only their degree and the number, $\tilde{e}$, of links in the component that contains it which are not incident to it. The model is closest to Blume et al. (2013). The difference is that in our model the likelihood of a node initially defaulting to start contagion *depends* on the network.

Proposition 1. *The expected payoff of a firm with degree d and whose component has $\tilde{e}$ other links than the d links of the firm is given by* $\tilde{U}(d, \tilde{e}) := d\alpha^{2(\tilde{e}+d)}\theta$.

The functional form of $\tilde{U}(d, \tilde{e})$ makes clear that holding $\tilde{e}$ fixed, the benefit from d increases up to a point and then declines. The benefit declines in $\tilde{e}$ the number of indirect links. In the sequel it is more convenient to write the firm's payoff as $U(d, e) := d\alpha^{2e}$ where $e = \tilde{e} + d$ is the total number of links in the relevant component.

2.2 Stability and the Periphery

As network formation occurs prior to the realization of shocks, firms evaluate each network (N, E) with respect to their expected payoffs in the continuation. The payoffs in the contin-

[3] We recall some standard definitions from graph theory. i and j are *adjacent* if they have a link. Adjacent firms are counterparties in our model. If i and j are adjacent, we say i and $\{i, j\}$ are *incident*. i and j are *connected* if one can be reached from the other via a sequence of links. The *component* of i is the set of all firms that are connected to i and all links between these firms. A *clique* is a collection of firms with a link between each pair. A *disjoint clique* a component which is a clique. The *order* of a clique is the number of firms in the clique. A $(d+1)-$*clique* is disjoint clique of order $d+1$. Note that each firm in a $(d+1)$-clique has degree d.

[4] Elliott et al. (2014) call this a network of full integration. They study the trade-off between integration and diversification but do not characterize the network formed. Although the models are different, our paper can be seen as the network formed in the corner case of full integration and no diversification.

uation are given by $U(d,e)$. There is an upper bound $\tilde{d} \leq n-1$ such that firms cannot form more than $\tilde{d}$ links. Consider a candidate network (N,E) and a subset of firms $N' \subset N$. A *feasible deviation* by N' allows firms in N' to *simultaneously*

1. add absent links within N',
2. delete any link incident to at least one vertex in N'.

A *profitable deviation* by $N' \subset N$ is a feasible deviation that strictly improves the expected payoff of each member of N'. (N,E) is *bilaterally stable* if there are no profitable deviations for any $N' \subset N$ with $|N'| \leq 2$. (N,E) is called a *group stable* if there is no profitable deviation for any $N' \subset N$.[5]

Bilateral stability is a permissive solution concept because such networks are immune to deviations by pairs only. Proposition A1 in Appendix A shows that they consist of disjoint cliques of varying size (see Blume et al. (2011), Elliott et al. (2020) and Erol (2016) for similar results). To see why cliques are essential for stability, suppose three firms $\{i,j,k\}$ and also that the links $(i,j),(j,k)$ are present whereas $\{i,k\}$ is not. Consider firms i and k. If they cut the links (i,j) and (j,k) and replace them with (i,k) their degrees remain unchanged. However, they are now part of a component with strictly fewer links. Hence, their expected payoffs can only increase. Therefore, stability induces connected firms to be counterparties of each other which implies that the network must consist of disjoint cliques.

As there can be multiple bilaterally stable networks, we focus on group stability, which, when they exist, are unique (up-to labeling). The expected payoff of a firm in a $(d+1)$-clique of firms is

$$V(d) = U(d,(0.5)d(d+1)) = d\alpha^{d(d+1)}\theta.$$

Let $d^* = \arg\max_{d \leq \tilde{d}} V(d)$ and call this the optimal degree. Note that V is strictly log-concave (so single-peaked) and d^* is generically well defined. In fact, $d^* = \min\{\lfloor x^* \rfloor, \tilde{d}\}$ where x^* satisfies $\frac{1}{x^*} + \alpha^{2x^*} = 1$. While x^* cannot be determined explicitly, it is the case that $y^* - 1 \leq x^* \leq y^* + 0.25$ where $y^* = \sqrt{(-2\ln(\alpha))^{-1}}$.

Theorem 1. *Suppose that n is divisible by d^*+1. There exists a unique (up-to permutation) group stable network and it consists of disjoint (d^*+1)-cliques.*

The network formed is illustrated in Figure 2. In the absence of the divisibility assumption group stable networks need not exist. However, as shown in Appendix A, a group stable

[5] In Dutta and Mutuswami (1997) and Erol (2016) this solution concept is called strongly stable. Farboodi (2015) calls this solution concept group stable. We think the second more evocative. Jackson and Van den Nouweland (2005) considers a stronger notion that rules out deviations by any coalition V that weakly improves the payoff of all members, and strictly improves the payoff of at least one member.

network, if it exists, must consist of as many disjoint $(d^* + 1)$-cliques as possible, and the remainder firms are also in a disjoint clique. Existence fails when the number of firms in the remainder falls below a threshold.[6] One can eliminate this possibility by restricting the size of a deviating coalition. We think it simpler to invoke the divisibility assumption. It is not unusual in the network formation literature, see, for example Elliott et al. (2020).[7]

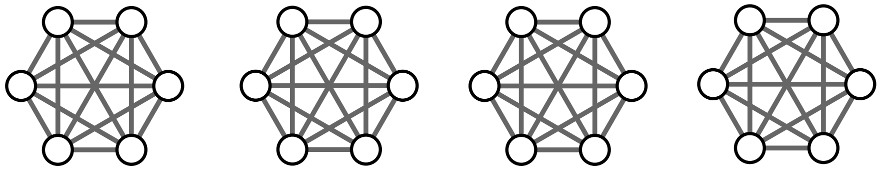


Figure 2: Structure of the periphery under group stability

Group stable networks need not be Pareto efficient, and even if they are, they need not be utilitarian efficient (maximize sum of expected payoffs of firms among all realized networks).[8] Within the context of systemic risk, Farboodi (2015) in a different model, finds that group stable networks are utilitarian inefficient. Interestingly, this is not the case here.

Consider a connected component with e links. A firm in the component with degree d enjoys an expected payoff of $d\alpha^{2e}\theta$. Therefore, the sum of payoffs of firms within the component is $2e\alpha^{2e}\theta$. It follows that the problem of finding an efficient network devolves into two parts: how to partition firms into components and how many links to put into each component.

Theorem 2. *Suppose that n is divisible by $d^* + 1$. A network is utilitarian efficient if and only if it is group stable.*

Observe first that only the number of links in a component matters for the efficiency of the component. Therefore, conditional on a component having e links, we want it to have as few nodes as possible so as to have as many components as possible. Thus, components should be as dense as possible, i.e., nearly cliques. The particular functional form of the expected payoffs ensures that no link is absent from a component. Conditional on disjoint cliques, the efficient size of a clique should maximize the average payoff of firms in the clique. Average payoff in a disjoint clique with $d + 1$ firms is $d\alpha^{d(d+1)}$, and so the efficient order of the clique

[6]This problem does not arise with bilaterally stable networks because cliques are not all of the same size.

[7]We can replace the divisibility assumption in Theorem 1 with a variant of farsighted stability at the cost of uniqueness. For example, weaken strong stability by precluding only the profitable deviations after which the resulting network does not involve any profitable deviations by a coalition that includes any of the previous deviators. This ensures that the networks in Proposition A2 are strongly stable.

[8]Under a stronger notion of stability, see Jackson and Van den Nouweland (2005), that excludes Pareto improvements, group stability implies Pareto efficiency.

is $d^* + 1$. Note that, all bilaterally stable networks other than the group stable network are inefficient.

Absent the divisibility assumption, a utilitarian efficient network will coincide with the group stable network (if it exists) and consist of as many disjoint $(d^* + 1)$-cliques as possible, with the remaining firms also in a disjoint clique.

2.3 Peripheral Systemic Risk and the Volatility Paradox

Peripheral systemic risk The disjoint clique structure observed here will be present in the periphery of the larger model. Accordingly, the systemic risk that we study in this section corresponds to the peripheral systemic risk of the larger model, i.e., the risk that most peripheral firms default because of contagion initiated by shocks to the periphery firms. Later, when we study the full model of the core-periphery, we will be interested in central systemic risk, i.e., the risk that most peripheral firms default because of contagion initiated by shocks to the core.

The network (periphery) formed consists of many disjoint cliques. Each clique is subject to contagion within itself. There is an $\alpha^{d^*(d^*+1)}$ probability that all shocks to a clique are good, therefore no firm in the clique defaults. Otherwise, all firms in the clique default. Since cliques are disjoint, the defaults of cliques are independent events. This allows us to pin down the exact distribution of the number of defaults, which is reported in Appendix B.1.

Systemic risk is typically viewed as the tail event that a large fraction of the system is damaged. For simplicity, we focus on the extreme tail event that *all* peripheral firms default, and call the probability of this event systemic risk. Normalized systemic risk will refer to systemic risk to the power n^{-1}. Appendix B.1 derives the distribution of the number of defaults and examines its first and second moment as other measures of systemic risk. The probability that all firms in a disjoint clique default is $1 - \alpha^{d^*(d^*+1)}$. Since there are $\frac{n}{d^*+1}$ many components, normalized systemic risk (in the absence of the core) is

$$\left(1 - \alpha^{d^*(d^*+1)}\right)^{(d^*+1)^{-1}}. \tag{1}$$

Peripheral volatility paradox Of interest is how systemic risk changes in response to α, which is the model parameter that captures the exogenous and fundamental risk in the economy. On one hand, for a fixed network, as α increases, projects become less risky, contagion is triggered less often, and so the economy is in a fundamentally better state. This reduces any reasonable measure of systemic risk for a fixed network. On the other hand, for fixed α, more interconnected networks create more contagion in our model. This increases systemic risk. In the endogenous network that is formed, as α increases, d^* increases, and the network becomes more interconnected. This is depicted by the solid line in Figure 3*a*. A priori it is unclear whether the reduction of risk due to higher α or the increment in risk due to higher interconnectedness dominates.

In Figure 3*b*, the solid line is the normalized systemic risk. The kinks arise from the integrality constraint on d^*. Systemic risk of the group stable network increases with α (modulo the non-monotonicity due to discreteness of the model).[9] We call this the *peripheral volatility paradox*.

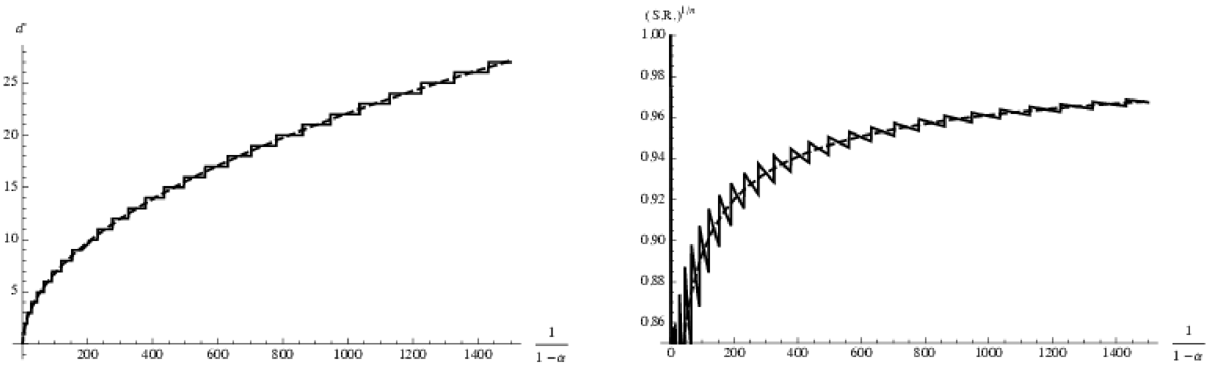


Figure 3*a*: Degree d^* in α

Figure 3*b*: Systemic risk in α

Figure 3: Degree d^* and systemic risk

As we are unable to derive a closed-form expression for systemic risk as a function of α (due to discreteness), we derive tight lower and upper bounds on it. These bounds grow and eventually merge with each other as α increases. This shows that the graphs in Figure 3 are not a trick of the eye.

Let x^* be the unique positive solution to $\alpha^{2x} + \frac{1}{x} = 1$. Let $\tilde{A} = \{\alpha : x^* \in \mathbb{N}\}$ and denote by $\mathbf{e}$, Euler's constant. Note that $\tilde{A}$ is a countably infinite set of points with a unique limit point, 1.

Proposition 2. *If we relax the constraint $d^* \leq \tilde{d}$, then, $d^* = \lfloor x^* \rfloor$ for $\alpha \notin \tilde{A}$ and $d^* \in \{x^* - 1, x^*\}$ for $\alpha \in \tilde{A}$. The probability that a given firm defaults lies in $[1-\alpha^{(x^*-1)x^*}, 1-\alpha^{x^*(x^*+1)}]$. The lower bound is increasing and the upper bound is decreasing in α, both converging to $1-\mathbf{e}^{-0.5}$ as α approaches 1. Normalized systemic risk is between $\left(1-\alpha^{x^*(x^*-1)}\right)^{(x^*)^{-1}}$ and $\left(1-\alpha^{x^*(x^*+1)}\right)^{(x^*+1)^{-1}}$. Both bounds are increasing and converge to 1 as α approaches 1.*

[9] The same holds true for the variance of the number of defaulting firms.

The bounds on the individual firm default probabilities and systemic risk are tight in the sense that they are attained on $\tilde{A}$. As α increases from between two consecutive values in $\tilde{A}$, the actual probabilities fall continuously from the upper bound to the lower bound since d^* is fixed but α increases. Hence, the actual default probability of a firm oscillates between the bounds and around $1 - \mathbf{e}^{-0.5}$ as α increases, and approaches $1 - \mathbf{e}^{-0.5}$ as α approaches 1. Actual normalized systemic risk oscillates between the bounds and converges to 1 as α approaches 1. The oscillations are due to discreteness. Proposition 2 allows us to consider a proxy for optimal degree $(-2\ln(\alpha))^{-0.5} \in [x^* - 1, x^*]$ and a proxy for the default probability of a firm $1 - \mathbf{e}^{-0.5}$.[10]

From Proposition 2 we know that for a given clique, there is a roughly $1 - \mathbf{e}^{-0.5}$ probability that all firms in the clique default and a roughly $\mathbf{e}^{-0.5}$ probability that no firms in the clique default. Therefore, normalized systemic risk is approximately

$$\left(1 - \boldsymbol{e}^{-0.5}\right)^{\sqrt{-2\ln[\alpha]}}$$

which is the dashed line Figure 3*b*.

Note, systemic risk is increasing in α, (modulo oscillations around the limit due to discreteness) illustrating the peripheral volatility paradox. As α increases, the clique size grows unboundedly. However, the default probability of each clique is bounded away from zero (oscillating around the limit due to discreteness). So, the probability that all cliques fail *increases*. This becomes apparent when we use the approximate values. The default probability of a clique is roughly constant, but the number of cliques decrease, which increases systemic risk.[11]

The corresponding estimate for systemic risk is then

$$\left(1 - \boldsymbol{e}^{-0.5}\right)^{n\sqrt{-2\ln[\alpha]}}$$

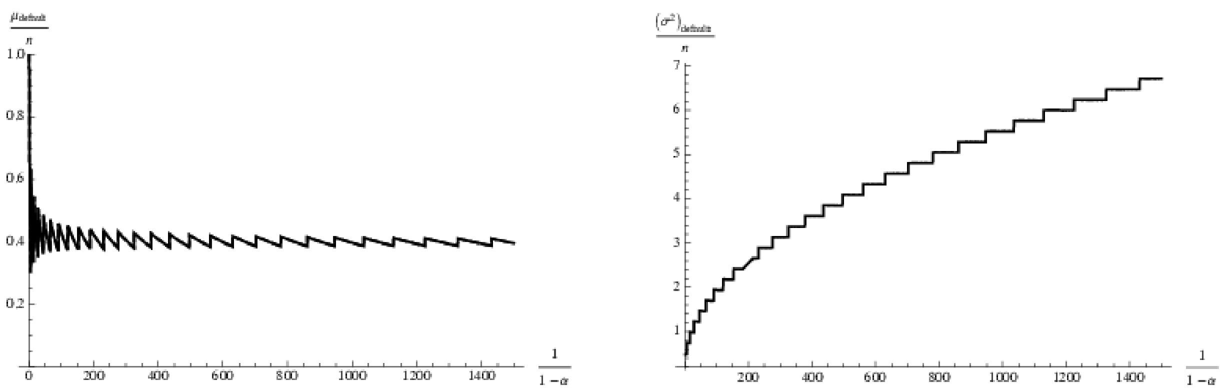
which is the dashed line Figure 3*b*. Note that the estimated systemic risk is increasing in α, yielding the peripheral volatility paradox. For fixed n, as α increases, the clique size grows,

[10] There are other reasonable approximations. For example, one can take $\arg\max_{y\in\mathbb{R}} y\alpha^{y(y+1)} = -\frac{1}{4} + (\frac{1}{16} - (2\ln(\alpha))^{-1})^{0.5}$ as an approximation for d^* and $\alpha^{y(y+1)}$ for the default probability of a firm. Going forward, we will alert the reader to whether the selection of the approximation has a material effect on the insights.

[11] Recall that there is an upper bound $\tilde{d} \leq n - 1$ on the number of links a firm can have. Once α becomes too large and hits $\left(1 - \tilde{d}^{-1}\right)^{(2\tilde{d})^{-1}}$, d^* becomes $\tilde{d}$ and the cliques cannot get any larger. Systemic risk cannot get any larger and starts decreasing. $\tilde{d}$ could very well be equal to $n - 1$, meaning that the cliques would grow until becoming the complete network.

and the number of cliques decreases. There are a smaller number of cliques, each of which has a constant probability of failing. So, the probability that all cliques fail *increases*.[12,13]

A closer examination of the systemic risk as the probability of a good shock increases reveals that the mean number of defaults is bounded away from zero (oscillating around the limit due to discreteness) yet, the *variance of the number of defaults increases* as well as systemic risk. The mean and variance are pictured in Figure 4. The added risk due to the change in the sizes of cliques $d^* + 1$ offsets the reduced risk from increasing the probability of a good shock α. The probability that a clique fails is bounded away from zero, remaining roughly constant close to the limit. However, as the cliques grow, the defaults of single firms become more correlated. Therefore, the variance of defaults increases. The heightened correlation increases the probability of the tail event, i.e., systemic risk.



Mean Variance

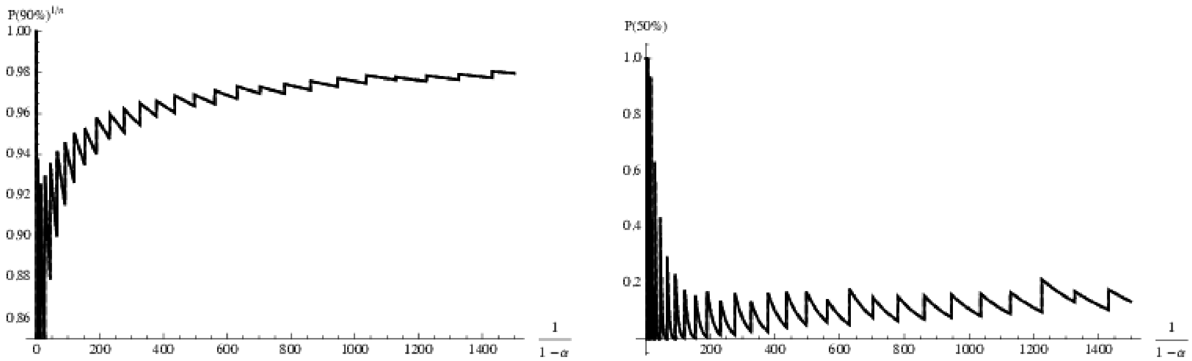
Figure 4: Mean and variance of the number of defaults

In our framework, the variance of defaults and systemic risk increase in α because of endogenous correlations arising from network formation, summarized by the clique structure and d^*. This yields a network-rooted version of the volatility paradox propounded by Brunnermeier and Sannikov (2014). The key idea is that better fundamentals encourage firms to take on more investments, but these investments are the conduits of contagion above and beyond simply being the exogenous sources of risk. As such, investment returns are endogenously correlated through network formation, giving rise to volatility paradox.

[12]Notice that if one increases n, systemic risk actually *decreases* because the probability that more cliques fail is smaller than the probability that few cliques fail, given that the failures of cliques are independent events with probability $1 - \boldsymbol{e}^{-0.5}$.

[13]Recall that there is an upper bound $\tilde{d} \leq n - 1$ on the number of links a firm can have. Once α hits $\left(\frac{\tilde{d}-1}{\tilde{d}}\right)^{\frac{1}{2\tilde{d}}}$, d^* becomes $\tilde{d}$ and the cliques cannot get any larger. Systemic risk cannot get any larger and starts decreasing. $\tilde{d}$ could very well be equal to $n - 1$ meaning that the cliques would grow until becoming the complete network.

The volatility paradox holds for other measures of systemic risk as well.[14] We illustrate this for some measures here with figures. Figure 4 shows the mean and variance of the number of defaults. Figure 5 shows the probability 90% and 50% of firms defaults. Figure 6 shows the mean and variance of welfare, measured as the sum of firms' payoffs. In all these cases, variances and tail risks are roughly increasing in α for the same reason as systemic risk.



Normalized prob. that > 90% default　　Prob. that > %50 default

Figure 5: Probability that at least a certain fraction of firms default

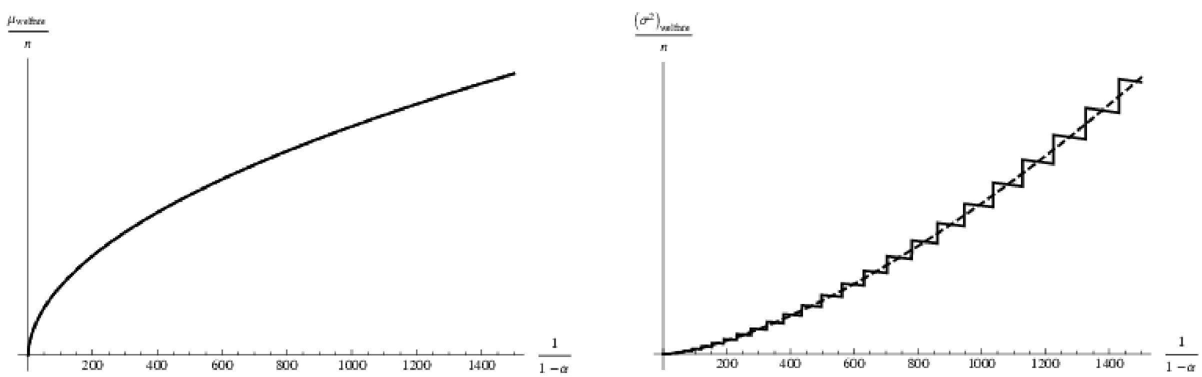


Figure 6: Welfare mean and variance

Increased correlation by itself does not justify an increase in variance or tail risk. The elasticity of substitution between link benefits (d here) and the continuation probability ($\alpha^{d(d+1)}$ here) also matter. In our model, increased risk-taking and reduced fundamental risk, $1-\alpha$, somewhat offset each other. Even when the fundamental risk $1-\alpha$ approaches zero, firms take on a large number of projects that keeps the default risk of a clique bounded away from zero. Therefore, connectivity of the network, i.e., the reduction in the number of cliques, is the primary determinant of variance and tail risk. However, if the endogenous risk of a clique fell faster than the number of cliques, (and so approach zero) there would

[14] Appendix B.1 contains a brief analysis.

potentially be no volatility paradox. Appendix B.2 further illustrates that the volatility paradox in our model is a consequence of contagion, not of the specific parameters.

2.4 Popcorn vs. dominoes

Earlier, we drew attention to a debate about whether interconnectedness of firms is a significant contributor to systemic risk. An alternative theory is that the risk faced is systematic, i.e., popcorn. We model the popcorn story as perfect correlation in the link shocks. Suppose the probability that all links are good is σ and the probability they are all bad is $1-\sigma$. Now there is no risk of contagion, and all risk is due to common exposures. The next two results establish that "popcorn" and "dominoes" are observationally equivalent in terms of network structure but differ in probability of system-wide failure.

Proposition 3. *Suppose that n is divisible by $\tilde{d}+1$. Under 'popcorn', the unique group stable (and the unique bilaterally stable) network consists of disjoint cliques of order $\tilde{d}+1$.*

When firms face systematic risk (popcorn) rather than systemic risk (dominoes), they form highly interconnected networks in order to reap the benefits of trade. Under popcorn, all firms are in the same boat, and the major risk is due to common exposure rather than contagion. There is no need to refrain from forming too many links. If the underlying common shock is good, then firms get higher returns by having more links. If the underlying common shock is bad, eliminating some links cannot change the outcome.

The distinction between dominoes and popcorn is related to what can be estimated about the probability of system-wide failure from network structure. Suppose the probabilities of good shocks are $\alpha \in (0,1)$ and $\sigma \in (0,1)$ respectively. If the realized network consists of $(d+1)$-cliques where $d < \tilde{d}$, it can be inferred that the risk is dominoes. The realized degree pins down the values of α and the systemic risk. In particular, systemic risk is given by Equation (1), which is roughly $(1-e^{-0.5})^{\frac{n}{d+1}}$. On the other hand, if the realized network consists of $(\tilde{d}+1)$-cliques, only a partial inference of the correlation structure and the probability of system-wide failure can be drawn.

Theorem 3. *Given, a network consisting of $(\tilde{d}+1)$-cliques. Conditional on the popcorn world, the probability of system-wide failure can range from 0 to 1. Conditional on the dominoes world, the probability of system-wide failure can range from 0 to*

$$\left(1-\left(1-\tilde{d}^{-1}\right)^{\frac{\tilde{d}+1}{2}}\right)^{\frac{n}{\tilde{d}+1}}.$$

If $\alpha < (1-\tilde{d}^{-1})^{(2\tilde{d})^{-1}}$, then, the upper bound, $\tilde{d}$, on the optimal degree does not bind. If $\alpha > (1-\tilde{d}^{-1})^{(2\tilde{d})^{-1}}$ and the upper bound on degrees binds, the probability that a clique defaults is $1-\alpha^{\tilde{d}(\tilde{d}+1)}$. Combining this with the threshold for α that generates $(\tilde{d}+1)$-cliques gives us an upper bound on systemic risk.

Note that the upper bound is smaller than 0.5 for all $n \geq 6$ and $\tilde{d} \leq n-1$. Therefore, an outside observer who observes an endogenously formed dense network and believes it is the dominoes world would think that probability of system-wide failure is smaller than 0.5, whereas under the popcorn world, probability of system-wide failure can be arbitrarily close to 1. Similar results hold for all measures of systemic risk considered in Appendix B.1.

In sum, upon observing a dense network but not the correlation structure, one cannot accurately upper bound the probability of system-wide failure. Mistakenly believing that the risk is systemic rather than systematic can lead to underestimating the probability of system-wide failure.

3 The Core

3.1 The Model of the Core-Periphery

Here we extend the model in two dimensions. First, we introduce node shocks: shocks that directly impact firms rather than links. They model idiosyncratic operational firm costs. Contagion can be triggered by node shocks, which introduces another source of systemic risk. Second, we introduce contagion-resilient firms: their main risk of defaults stems from node shocks. Section 4 contains an interpretation.

Contagion resilient firms don't default when their incident links are hit by a bad shock or when their counterparties default (Assumption 2 below), but they do when subject to a bad node shock. If a bad node shock is unlikely (Assumption 3 below), such firms act as a 'fire break' against contagion. This makes contagion resilient firms attractive counterparties to all firms, generating a core-periphery structure. In Section 4, we show how such contagion resilient firms can emerge endogenously from firms that are initially ex-ante identical.

There are two types of firms: *contagion-resilient* types and *normal* types. There are m contagion-resilient firms denoted $M = \{1, ..., m\}$ and n normal firms denoted $N = \{m+1, ..., m+n\}$. All pairs of firms can form links. As before, we assume that firms form group stable networks given the expectation of their payoffs in the continuation. After the network is formed, links receive good or bad shocks. Moreover, each firm i receives a *node shock* which is good or bad. A firm with a bad node shock defaults and earns 0.

For normal firms, the probability of a good link shock is α and a good node shock is ζ. A normal firm $i \in N$ with d_i counterparties, f_i many defaulting counterparties, and b_i many bad links that receives a good node shock, has payoff $u_i = d_i\theta - \beta f_i - \gamma b_i$ if it continues. If $u_i \geq 0$, i continues and enjoys u_i. Otherwise, i defaults and earns 0. Assumption 1 still holds for normal firms.

For contagion-resilient firms, the probability of a good link shock is α' and a good node shock is ζ'. A contagion-resilient firm $i \in M$ that enjoys a good node shock, with d_i many counterparties, f_i many defaulting counterparties, and b_i many bad links has payoff $u'_i = d_i\theta' - \beta' f_i - \gamma' b_i$ if it continues. If $u'_i \geq 0$, i continues and gets u'_i. Otherwise i defaults and gets 0. The difference between contagion-resilient and normal firms is encapsulated in the following assumption.

Assumption 2. *$\theta' > \gamma' + \beta'$.*

Assumption 2 ensures that a contagion-resilient firm will not default because of losses caused by a shock to an incident link or counterparty losses.

Therefore, the only way a contagion-resilient firm can trigger a contagion is when it is subject to a bad node shock. In fact, the model allows $\zeta' < \zeta$, so that contagion-resilient firms can be more vulnerable to node shocks than normal firms. In the rest of the paper, contagion-resilient firms will simply be called *resilient* firms, but bear in mind that such firms can trigger contagion more often than normal firms.

Under Assumption 2 we can pin down the expected payoffs of firms in a given network as follows. Call a path on the network a *normal path* if it consists of normal firms only.[15] For $i \in N$, consider the maximal rooted subtree rooted at i. This is the acyclic subgraph that contains i, the union of all normal firms which can be reached by normal paths from i, and the links in those paths. Denote the number of normal firms in this subtree by d_i^N. Denote by d_i^M the number of resilient firms that are counterparties of the d_i^N normal firms in the subtree rooted at i. Let e_i^N be the number of links in the subtree incident to these d_i^N firms. Denote by e_i^M the number of links in the cut set that separates the subtree from the rest of the network.

Note that the probability that a normal firm i does not default is given by

$$p_i := \zeta^{d_i^N} \zeta'^{d_i^M} \alpha^{2e_i^N + e_i^M}. \tag{2}$$

[15]The trivial path from $i \in N$ to itself is also a normal path.

Proposition 4. *The expected payoff of $i \in N$ is $u_i = d_i \theta p_i$. The expected payoff of $i \in M$ is*

$$u_i = d_i \zeta' \left(\theta' - \alpha'\gamma' - \zeta'\beta'\right) - \left(\sum_{j \in N \cap D_i} \left(p_j - \zeta'^2\right)\right) \beta'.$$

3.2 Stability and Efficiency of the Core-Periphery

The final piece needed to determine the group stable network is that resilient firms do not have significantly more idiosyncratic risk than normal firms, and so resilient firms are preferred counterparties with respect to the probability of triggering contagion.

Assumption 3. *Resilient firms' node risk is not too high compared to normal firms: $\zeta' > \alpha\zeta$.*

This does not preclude $\zeta > \zeta'$, so resilient firms can actually be *riskier* in terms of node shocks. If $\zeta' = \zeta$, this assumption is redundant.

Group stable networks When a normal firm i forms a link with a normal firm j, it must consider three risks aside from contagion through j. First, j can suffer a bad node shock with probability $1 - \zeta$. Second, j can suffer a bad link shock at link $\{i, j\}$ with probability $1 - \alpha$. Third, i can suffer a bad link shock at link $\{i, j\}$ which has probability $1 - \alpha$.

If i forms a link with a resilient firm k, it must consider only two such risks. First, k can suffer a bad node shock with probability $1 - \zeta'$. Second, i can suffer a bad link shock at link $\{i, k\}$ with probability $1 - \alpha$. While k can suffer a bad link shock at link $\{i, k\}$ with probability $1 - \alpha'$, this does not make k default, so it is not a risk for i.

Since $\zeta'\alpha > \zeta\alpha^2$, resilient firms are preferred counterparties, although it is possible that $\zeta' < \zeta$. Set $d_N^* = \arg\max_{d \geq 0} (m + d) \left(\alpha^m \zeta\right)^d \alpha^{d(d+1)}$ and $d_M^* = \arg\max d \left(\alpha\zeta\right)^d = \left\lfloor \frac{1}{1 - \alpha\zeta} \right\rfloor < m$. Note that if $d_M^* \leq m$ then $d_N^* = 0$. We confine ourselves to group stability here and report results on bilateral stability in the appendix.

Theorem 4. *Suppose that n is divisible by $d_N^* + 1$.*

(Case of small core) If $d_M^* > m$, a network is group stable if and only if it is a core-periphery of the following form:

1. *(Core) Each resilient firm is counterparties with all resilient firms and all normal firms.*

2. *(Periphery) Each normal firm is counterparties with all resilient firm and some normal firms. Excluding resilient firms and their incident links, normal firms form disjoint cliques of order $d_N^* + 1$ among themselves.*

(Case of large core) If $d_M^* \leq m$, a network is group stable if and only if it is a core-periphery of the following form:

1. *(Core) Each resilient firm is counterparties with all resilient firms and some normal firms.*
2. *(Periphery) Each normal firm has d_M^* resilient counterparties and no normal counterparties.*

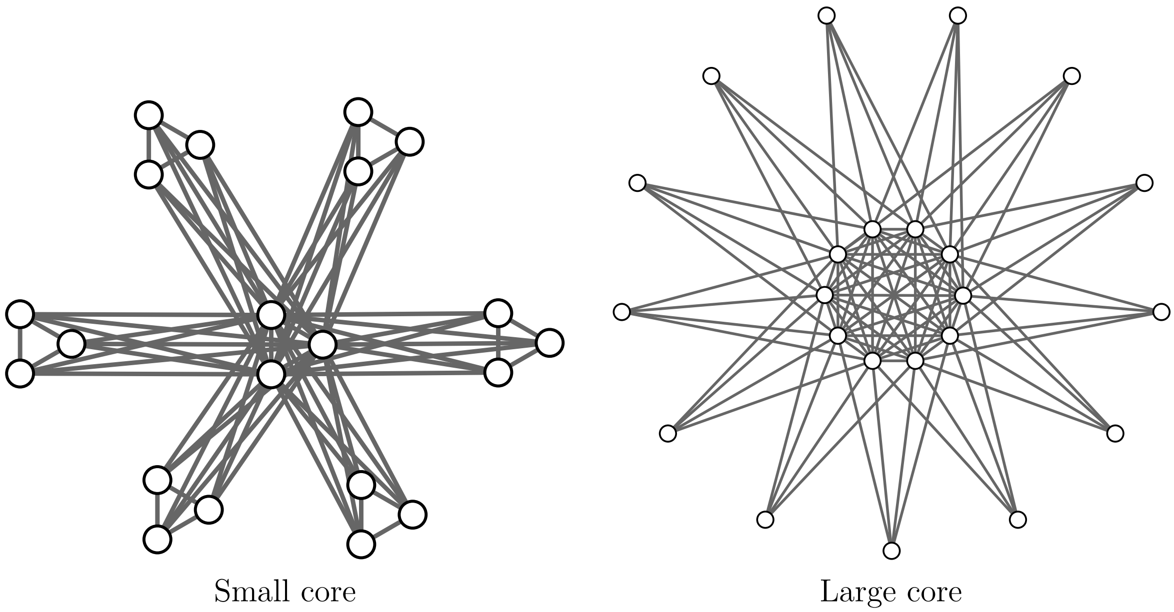


Figure 7: Structure of group stable networks

The possible group stable networks are displayed in Figure 7. Assumption 2 and Assumption 3 make resilient firms preferred counterparties. The only risk they impose on their counterparties arises from their node shock. As $\zeta' > \alpha\zeta$ this makes a resilient firm's node shock less of a risk than a normal firm's combined risk from the node shocks and the link shock. This does not mean that linking with the resilient firms entails no risk.

Group stability of the cliques in the periphery is a consequence of Theorem 1. Group stability of the links to the core is a consequence of resilient firms being preferred counterparties, despite the fact that connecting to the core still entails counter-party risk. For completeness, Theorem 3 in the appendix contains a discussion of the structure of bilaterally stable networks.

Efficiency In the model of the periphery with homogeneous firms, the utilitarian efficient network coincides with the group stable one. In the model of core-periphery with heterogeneous firms, firm payoffs differ across the core and the periphery and group stability and utilitarian efficiency no longer coincide. While the efficient network can be core-periphery, it need not coincide with either the small or large core that characterizes group stability.

Theorem 5. *Suppose that θ', γ' are sufficiently large, i.e., much larger than θ. The unique utilitarian efficient network is the core-periphery in which*

1. *each resilient firm is counterparties with all resilient firm and all normal firms,*
2. *and each normal firm is counterparties with all resilient firms and no normal firms.*

The idea is simple. Since θ' is large enough, efficiency dictates that core firms have links with all firms. Since γ' is large enough, it is efficient to lower the default risk for normal firms so as to reduce the externality they impose on core firms. This is achieved by not having any links between peripheral firms.

3.3 Central Systemic Risk and Volatility Paradox

Central systemic risk Linking to the core that consists of resilient firms is not risk-free because resilient firms are still subject to bad node shocks. For example, when $d_M^* \geq m$, if a resilient firm at the core suffers a bad node shock, it drives *the entire periphery* into default. Thus, the core is the major source of systemic risk in the economy. We call the probability of system-wide failure due to shocks to firms in the core *central systemic risk*. There is still peripheral systemic risk in the economy, which is the probability of the event that all cliques of peripheral nodes fail due to shocks to the peripheral firms.

Figure 8 illustrates the probability distribution of the number of defaults. The tail event in which most firms fail has high probability because if a resilient firm in the core suffers a bad node shock and defaults, it drags many normal firms in the periphery into default. Normal firms' risk of default becomes highly correlated through the node shocks of the core.

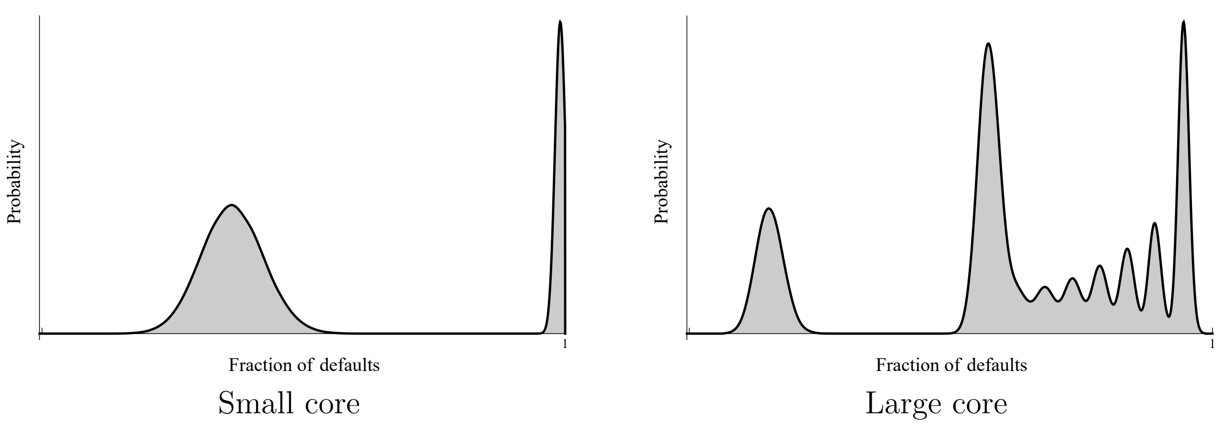


Figure 8: Probability distribution of the number of defaults

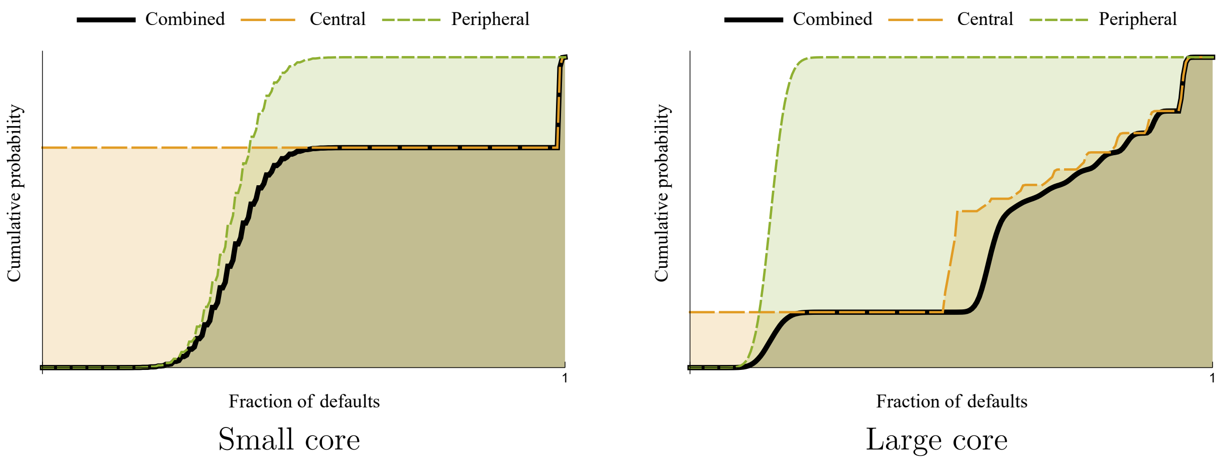


Figure 9: Cumulative probability distribution of the number of defaults with respect to sources of systemic risk

A closed-form expression for the PDF of the distribution of the number of defaults can also be determined. We have shown how this can be done for the periphery in Appendix B.1. One can incorporate the core similarly. We skip this for brevity and focus on the tail event that all firms in the periphery fail, i.e., systemic risk. In the case of a small core, all peripheral firms have links with all core firms. One bad node-shock to the core causes all peripheral firms into default. If all core firms enjoy good node-shocks, all peripheral firms default only if all cliques in the periphery get bad shocks of their own. The systemic risk is then given by

$$\underbrace{(1-\zeta'^m)}_{Central\,systemic\,risk} + \zeta'^m \underbrace{\left(1-\zeta^{d_N^*+1}\alpha^{\left(m+d_N^*\right)\left(d_N^*+1\right)}\right)^{\frac{n}{d_N^*+1}}}_{Peripheral\,systemic\,risk}. \tag{3}$$

In the case of a large core, there are multiple core-periphery networks that can be formed. In each of them, each peripheral firm has d_M^* core counterparties and no peripheral counterparties. Systemic risk can take various values depending on the overlap between the set of counterparties of each peripheral firm.[16] The simplest possibility is that all peripheral firms are counterparties with the same set of d_M^* core firms. In this case, systemic risk is given by

$$\underbrace{\left(1-\zeta'^{d_M^*}\right)}_{Central\,systemic\,risk} + \zeta'^{d_M^*} \underbrace{\left(1-\zeta\alpha^{d_M^*}\right)^n}_{Peripheral\,systemic\,risk}. \tag{4}$$

In what follows, we refer to (4) when we study systemic risk under the large core. In the

[16] Two sets X and Y have non-trivial overlap if $(X\backslash Y)\cup(Y\backslash X)\neq\emptyset$. When there are non-trivial overlaps between sets of counterparties of peripheral firms, no closed form expression for systemic risk exists. Nevertheless, we can pin down a value for systemic risk when there are no non-trivial overlaps between sets of counterparties of any pair of peripheral firms.

next section, we study the comparative statics of systemic risk to see whether the volatility paradox is present in the core-periphery network as well.

Central volatility paradox In Section 2.3, we identified and discussed a peripheral volatility paradox: as the probability of good link shocks α increase, the probability that all firms in the periphery fail via contagion initiated by shocks to the periphery (i.e., peripheral systemic risk) increases. This was specific to the periphery and cliques that emerge within the periphery. Under the core-periphery structure, similar arguments apply to the cliques within the periphery. As for the core, can we expect a different form of volatility paradox due to the shocks to the core? How does the probability that all firms in the periphery default (i.e. central systemic risk) due to shocks to firms in the core change as the probability of good shocks increase?

Consider the shocks to the links. First, we must understand how the structure of the network and the degree of peripheral firms change as α changes. Recall that resilient firms that occupy the core are preferred counterparties. Start with some small α such that $d_M^* \leq m$. Then peripheral firms want to be counterparties with only core firms. Thus $d_N^* = 0$ and so $d^* = d_M^*$. As α increases, eventually, $d_M^* > m$. At this point, there are not enough core firms, and $d^* = d_M^* + d_N^*$. In fact, d^* can be written compactly as $d^* = \min\{d_M^*, m\} + d_N^*$.[17] The way in which d^* changes as a function of α is portrayed in Figure 10*a*.

Next, we study how systemic risk changes in α. Beginning with some small α, the periphery is willing to form only a few links with the core: $d_M^* < m$. Recall that $d_M^* = \arg\max_d d\,(\zeta'\alpha)^d$. As α increases, so does d_M^*, and so the periphery is willing to form more links with the core. The risk internalized by a peripheral firm from adding one more link comes from both the node risk of the core firm, $1-\zeta'$, and the link risk of the newly added link $1-\alpha$. Peripheral nodes internalize the added default risk, $1-\zeta'\alpha$, and add links accordingly as α increases. The change in d_M^* keeps the risk $(\zeta'\alpha)^{d_M^*}$ constant (modulo the discreteness of the model). But central systemic risk increases with $1-\zeta'$ not $1-\zeta'\alpha$. d_M^* increases, so the central systemic risk $1-(\zeta')^{d_M^*}$ increases. As for combined systemic risk, we need to consider peripheral systemic risk too. In fact, $\alpha^{d_M^*}$ increases as α increases because $(\zeta'\alpha)^{d_M^*}$ is kept constant by the endogenous choice of d_M^*. Therefore, the peripheral systemic risk decreases because $1-\alpha^{d_M^*}$ decreases. As for the combined effect, note that the core is a more significant source of correlation in the default risk of the periphery. A Taylor approximation shows that the increase in central systemic risk dominates the decrease in peripheral systemic risk. This is plotted in Figure 10*b* for small to medium α. Combined systemic risk increases in α.

[17]It is worthwhile to note that $d^* = \arg\max_d d\zeta'^{\min\{d,m\}}\zeta^{d-\min\{d,m\}}\alpha^{d+(d-\min\{d,m\})^2}$.

Once α reaches a medium level, d_M^* becomes m, and there are no more firms in the core to form links with. The peripheral nodes start to link with each other. d_N^* starts growing. What happens to systemic risk once α is large enough that $d_M^* = m$? Now central systemic risk does not change anymore because there cannot be more connections to the core. Central systemic risk is set at $1 - (\zeta')^m$. Recall that $d_N^* = \arg\max_{d \geq 0} (m+d) (\alpha^m \zeta)^d \alpha^{d(d+1)}$. As α increases, d_N^* grows and the cliques in the periphery get larger. It turns out that the increment in d_N^* offsets the increase in α and peripheral systemic risk increases. In order to see why, note that $\zeta^{d_N^*} \alpha^{2d_N^*(d_N^*+m+1)}$ is made constant (modulo discreteness) by the endogenous choice of d_N^*, whereas the risk of a clique is $1 - \zeta^{d_N^*} \alpha^{d_N^*(d_N^*+m+1)}$. d_N^* increases as α increases. In turn, $\zeta^{d_N^*}$ decreases so that $\alpha^{2d_N^*(d_N^*+m+1)}$ increases. Then $\alpha^{d_N^*(d_N^*+m+1)}$ increases while $\zeta^{d_N^*} \alpha^{2d_N^*(d_N^*+m+1)}$ is constant. Therefore, $\zeta^{d_N^*} \alpha^{d_N^*(d_N^*+m+1)}$ decreases. Accordingly, the risk of a clique increases as α increases in addition to the increased risk from larger cliques and increasing correlation.

Peripheral systemic risk increases in α. Peripheral systemic risk is the only source of change in combined systemic risk as α increases because central systemic risk is constant. Therefore, combined systemic risk increases in α. This is plotted in Figure 10b for medium to large α. These comparative statics are summarized in Table 1.[18]

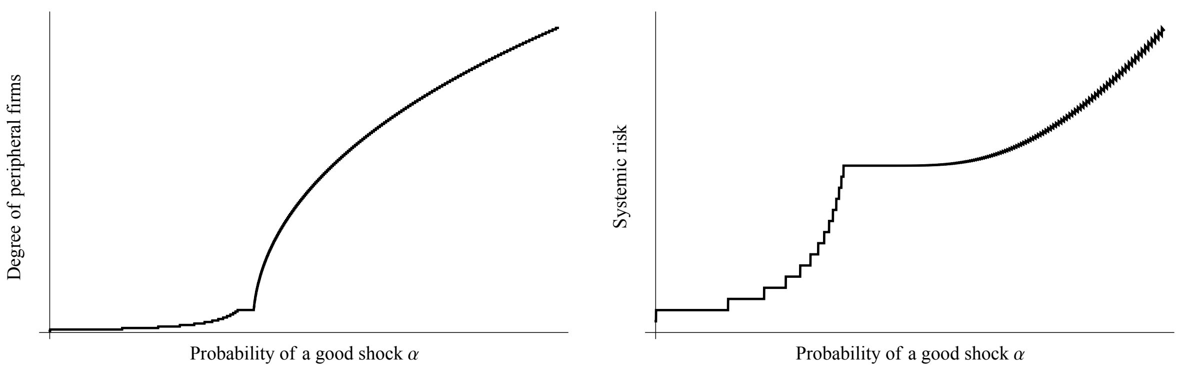


Figure 10a: Degree of normal firms in α Figure 10b: Systemic risk in α

Figure 10: Volatility paradox

Changing the probability of good node shocks impacts the network as well. However, the network never switches from $d^* \leq m$ to $d^* > m$ by changing ζ or ζ'. Whether the network is in the "small core" region or the "large core" region is determined by α. Thus we relegate the analysis of the node shocks to Appendix B.3.

[18]In fact, the probability that an individual firm fails is also increasing in α. Nevertheless, since all defaults in the periphery are highly correlated through the shocks to the core, systemic risk also increases.

Systemic risk	w.r.t.: source:	As α increases (link-shock) Central	Peripheral	Combined
$d^* \leq m$:		Increasing	Decreasing	Increasing
when α crosses a threshold, d^* crosses m				
$d^* > m$:		Constant	Increasing	Increasing

Table 1: Central and peripheral systemic risk in the probability of good link-shock α

4 Micro-foundation

In this section we micro-found the payoff structure used in Section 3 to show how a core-periphery network can emerge endogenously even when firms are ex-ante identical. This is in contrast to other explanations that rely on ex-ante *heterogeneous* agents, see for example Farboodi (2015), Erol (2016) and Wang (2016).

A link will correspond to a joint project funded by loans obtained from outside lenders. The project requires a monetary investment and costly effort from each counter-party before project maturity. Firms are free to drop any projects in their portfolio by not exerting effort. Dropped projects yield no return to the party that dropped them. Moreover, the return from a project is reduced if one's counter-party drops the project.

Firms that enjoy low-interest rates, hence high profit margins, correspond to the contagion resilient firms of Section 3 and therefore, will occupy the core. Firms that face a high-interest rate will end up occupying the periphery. A high-interest rate reduces the profit margins of a peripheral firm, making it unlikely to repay its debt. Therefore, peripheral firms have a higher default probability than non-peripheral ones. Hence, charging a high-interest rate to a peripheral firm is a self-fulfilling prophecy. An analogous argument will mean that core firms enjoy a low-interest rate. The resulting network and corresponding default probabilities will then be consistent with the interest rates charged.

We first describe firm payoffs and show they satisfy the assumptions of Section 3. There are k firms and each firm i has access to a line of credit L_i, at rate $R_i \geq 1$. Simultaneously with lending between lenders and firms, firms form a network of joint bilateral investments among themselves. P_i is firm i's per-project investment cost. Accordingly, firm i borrows $d_i P_i$ in total out of its credit line L_i to fund projects with d_i counterparties. In return, i promises to pay $R_i \geq 1$ per unit borrowed, but it is protected by limited liability. The number of links a firm can form is bounded by L_i/P_i. This bound corresponds to $\tilde{d}_i$ in the model.

After undertaking a project with firm j, firm i bears an uncertain management cost C^{ij} for the project, which can be either high ($\overline{C}_i$) or low ($\underline{C}_i$). The probability of enjoying the low management cost is α_i. These random costs correspond to the link shocks s_{ij} in the model.

Each firm also bears an operational cost κ_i, i.e., cost to continue their operations and stay in business, irrespective of their investments and project management costs. This cost can be either zero with probability ζ_i or take the value $K_i > 0$ with the complementary probability. This corresponds to the node shocks in the model.

Upon realization of the shocks, each firm i decides whether to *stay* in business at cost κ_i or *exit* the market to avoid incurring κ_i. If i stays, it also chooses which among the projects it has invested into *keep* and which to *drop*. In this case, if i drops the project with j, i receives nothing from the project, but also avoids the management cost C^{ij}. If i keeps the project with j, i incurs the management cost C^{ij}. i's return depends on the effort of j. If j also keeps the project, i gets revenue $\overline{B}_i$ from the project. If j drops the project, i gets revenue $\underline{B}_i$ from the project, where $\overline{C} < \underline{B}_i < \overline{B}_i$. In the event that i exits the market, all of its projects are dropped.

The assumptions needed to map this setup on to our model is that for all i, $\underline{B}_i - \overline{C}_i$ is positive and K_i is sufficiently large. In this case, firms with high operational costs (node shocks) exit, and drop all projects. For firms with low operational costs, it is a best response to either keep all projects or drop all of them. This is formalized in Proposition A4 in the appendix, and it can be seen simply as follows. If it is possible for i to get positive payoffs, it will keep all projects because each project has a positive surplus as $\underline{B}_i - \overline{C}_i > 0$. If it is impossible to obtain a positive payoff even after keeping all projects, then i prefers to drop all of its projects. This ensures 0 payoff because i's pecuniary payoff is protected with limited liability, whereas dropping the projects avoids effort costs. This micro-founds the modeling choice that firms have a binary action to continue (stay and keep all project) or default (exit, or stay but drop all projects). For each firm i set

$$\theta_i = \overline{B}_i - \underline{C}_i - P_i R_i,\ \beta_i = \overline{B}_i - \underline{B}_i,\ \gamma_i = \overline{C}_i - \underline{C}_i. \tag{5}$$

Then, i's ex-post payoff is

$$d_i\theta - f_i\beta - b_i\gamma$$

if it optimally continues, where f_i is the number of counterparties that default and b_i is the number of link shocks.

If R_i is sufficiently close to $P_i^{-1}(\overline{B}_i - \underline{C}_i)$, one link shock or one defaulting counterparty forces i into default. This is the case for peripheral firms. The counterpart of this parametric restriction is Assumption 1. On the other hand, if $R_i < P_i^{-1}(\underline{B}_i - \overline{C}_i)$, no number of link shocks or counterparty defaults can force i into default. This is the case for core firms. The counterpart of this parametric restriction is Assumption 2. Therefore, Sections 2 and 3 are a special case of this microfoundation with two types of firms, one with high interest rate and one with low interest rate. This is formalized in Proposition A5 in the appendix.

4.1 Core-periphery with ex-ante identical agents

We show that a core-periphery network will emerge in equilibrium when all firms are ex-ante identical. We suppress all subscripts except for the endogenous variables. Assume that the creditors of firm i consist of a unit mass of risk-neutral lenders, each of whom has L to lend to the firm. Denote by R_i the endogenous interest rate that firm i faces. The lenders are infinitesimal, so they take interest rates as given and decide whether to lend or not independently of each other.

Definition. *An **equilibrium** is defined as a collection of interest rates R_i one for each firm i and a network among k firms, such that the following two statements hold:*

- *Given the network formed, R_i takes a value that makes the lenders to firm i indifferent between lending or not.*
- *Given interest rates $\{R_i\}_{i\leq k}$, the network formed is group stable.*

Assumption M 1. $K > L\left(\frac{\overline{B}-\underline{C}}{P} - 1\right)$

This ensures that being subjected to a large operational cost is sufficient to force firms into default regardless of the interest rate. Under Assumption M1, interest rates must be at least ζ^{-1} to provide a premium for the operational risk.

Assumption M 2. $\frac{\underline{B}-\overline{C}}{P} > \frac{1}{\zeta}$

This ensures that a firm with the lowest possible interest rate ζ^{-1} faces no default risk other than operational risk (node shock). Assumption M2 is the counterpart of Assumption 2. Assumption M2 implies that $\underline{B} - \overline{C} > 0$. Then, it is a best response for each firm to either keep all its projects or default on all of them.

Assumption M 3. $\frac{\overline{B}-\underline{C}}{P} > \frac{1}{\zeta(\alpha\zeta)^{\left\lfloor\frac{1}{1-\alpha\zeta}\right\rfloor}} > \frac{\overline{B}-\underline{C}}{P} - \frac{\min\left\{\overline{B}-\underline{B},\overline{C}-\underline{C}\right\}}{L}$.

Assumption M3 concerns firms with a high interest rate, $\zeta^{-1}(\alpha\zeta)^{-\left\lfloor\frac{1}{1-\alpha\zeta}\right\rfloor}$. The leftmost inequality implies that such firms have positive expected payoff by forming links whereas the rightmost one implies that any link shock or counterparty default will result in the default of the firm. This is the counterpart of Assumption 1.

Assumption 3 has no counterpart here since firms are ex-ante identical.

Theorem 6. *Suppose that Assumption M1, M2, M3 hold and* $k > \left\lfloor \frac{1}{1-\alpha\zeta} \right\rfloor$*. Then, for each integral* m *between* $\left\lfloor \frac{1}{1-\alpha\zeta} \right\rfloor$ *and* k*, there exists an equilibrium such that* m *many firms face interest rate* ζ^{-1}*. The remaining* $n := k - m$ *firms face interest rate* $\zeta^{-1} (\alpha\zeta)^{-\left\lfloor \frac{1}{1-\alpha\zeta} \right\rfloor}$*. The network is core-periphery as described in Theorem 4, where the* m *firms that enjoy the low-interest rate are in the core, and the remaining* n *firms with high-interest rate are the periphery.*

4.2 Systemic Risk, Efficiency, and Policy

Here we compare the different equilibria in terms of their efficiency and systemic risk.

Theorem A1 in the appendix characterizes the equilibrium networks and we summarize it here. There are three types of equilibrium networks.

1. **Large-core equilibria**
   Peripheral nodes enjoy an interest rate $R = \zeta^{-1}\mathbf{e}$.[19] The size of the core is any $m \geq \overline{m}$ where $\overline{m}$ depends only on α, ζ. In these equilibria, each peripheral node has $\overline{m}$ links, all with the core.
2. **Medium-core equilibria**
   Peripheral nodes enjoy an interest rate $R < \zeta^{-1}\mathbf{e}$ and the core size is $m_2(R)$ where m_2 is a decreasing function with values in $[\underline{m}, \overline{m})$ and $\underline{m}$ depends only on α, ζ.
3. **Small-core equilibria**
   Peripheral nodes enjoy an interest rate $R < \zeta^{-1}\mathbf{e}$ and the core size is $m_1(R)$ where m_1 is an increasing function with value less than $\underline{m}$.

In medium-core and small-core equilibria, each peripheral firm is counterparties with all core firms.

Suppose a sufficiently large number of firms for a clean analysis. All else fixed, the systemic risk is roughly equal to the central systemic risk. In a large-core equilibria, each peripheral firm has a fixed number of counterparties $\overline{m}$ in the core and none in the periphery.

Large core equilibria differ only in the number of firms that are in the core. The interest rate is the same across all large core equilibria. The larger the core, the more efficient the network. The larger the core, the lower its systemic risk because the core-counterparties of the peripheral firms are more dispersed across the core. Hence, the default risk of peripheral firms becomes less correlated.

[19]In the formal result we focus on values α, ζ such that integrality problems do not arise. For these values we have $\zeta^{-1} (\alpha\zeta)^{-\left\lfloor \frac{1}{1-\alpha\zeta} \right\rfloor} = \zeta^{-1}\mathbf{e}$.

Among the medium-core and small-core equilibria, interest rates vary. Nevertheless, it is still the case that efficiency increases with the size of the core. Systemic risk, however, need not decrease. This has implications for interventions designed to increase the size of the core. In our model, this could be achieved by lowering the interest rate for peripheral firms. Assume that the network adjusts to the intervention as $m_1(R)$ and $m_2(R)$ would suggest. For medium-core equilibria, the size of the core *decreases* as the interest rate R faced by peripheral firms is reduced. This makes the core face less shocks, reducing systemic risk. On the other hand, for small-core equilibria, the size of the core *increases* with lower interest rates, increasing systemic risk. Thus, more efficient core-periphery equilibrium networks become more financially stable, and less efficient core-periphery equilibrium networks become less financially stable in response to expansionary monetary policies.

5 Conclusion

This paper introduced a model of endogenous network formation and systemic risk which furnishes three lessons. First, the network formed depends on the correlation between shocks to links. Misconceiving this correlation causes one to underestimate the probability of system-wide default. Second, even if the economy becomes fundamentally safer (the probability of good shocks increases), the probability of system-wide default can increase. Systemic risk increases because of the endogenous correlation of default risk that stems from network formation. Finally, in spite of the fact that the core-periphery structure facilitates systemic risk because the core with its dense interconnections encourages contagion, this does not prevent their emergence. In fact, while efficiency increases with the size of the core, systemic risk need not decline. This has implications for interventions designed to reduce systemic risk. Whether such interventions reduce systemic risk or not will depend upon the size of the core.

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Appendix A Proofs and Auxiliary Results

Proof of Proposition 1: expected payoffs

Proof. By Assumption 1, $d_i\theta \leq \tilde{d}\theta < \gamma$. Therefore, one bad shock forces a firm into default. Also $d_i\theta \leq \tilde{d}\theta < \beta$, so that one defaulting counter-party forces a firm into default. That is, either all shocks are good in the component and all firms continue their business, or at least one shock is bad and all firms default. There are e links in the component. For each link, two shocks, one for each incident firm, realizes. So $2e$ shocks realize in the component. There is an α^{2e} probability that the component survives in which case the firm with degree d has payoff $d\theta$. Otherwise, the firm defaults and earns 0. Thus, the expected payoff of a firm is $d\alpha^{2e}\theta$. □

Bilateral stability

Proposition A 1. *Any bilaterally stable network consists of disjoint cliques.*
There exist two numbers $\underline{d}$,$\overline{d}$, and a decreasing function ϕ with $\phi(\underline{d}) = \underline{d}$ such that $G = (N, E)$ is bilaterally stable if and only if G consists of disjoint cliques and

1. *either all cliques have orders between $\underline{d} + 1$ and $\overline{d} + 1$,*
2. *or only one clique has order less than $\underline{d}$, say $\underline{d}_G + 1 \leq \underline{d}$, and all other cliques have orders between $\phi(\underline{d}_G) + 1$ and $\overline{d} + 1$.*

There exists $\underline{\alpha} < 1$ such that for all $\alpha > \underline{\alpha}$, there is an $\underline{n}$ such that for all $n > \underline{n}$, there exists a bilaterally stable network.[20]

Proof. Take three firms $\{i, j, k\}$ and suppose the links $(i, j), (j, k)$ are present whereas $\{i, k\}$ is not. Consider firms i and k. If they cut the links (i, j) and (j, k) and replace them with (i, k) their degrees remain unchanged. However, they are now part of a component with strictly fewer links. Hence, their expected payoffs strictly increases their payoff. Therefore, all connected firms must be counterparties in bilaterally stable networks, meaning that all components must be cliques in bilaterally stable networks.

Now fix a network that consists of disjoint cliques. We consider all types of deviations by pairs from this network.

[20] $\underline{\alpha} < \alpha$ is needed to ensure $\underline{d}$ and $\overline{d}$ are not equal. $n \geq \underline{n}$ is needed to ensure that orders of the cliques can be arranged between $\underline{d}$ and $\overline{d}$ in way to sum up to n. Indeed, for small α, if n is odd, $\underline{d} = \overline{d} = 2$ and so the smallest clique has to be singleton, but $\phi(1) > 1$. So there does not exist a bilaterally stable network in that case. If n and α are not too small, the orders of cliques in a bilaterally stable network can be arranged in a way to sum up to n and we have existence.

Case 1: Unilateral deviation. Consider firm i and a deviation in which i cuts $d_i - y$ links and now has y links. There are $y + \frac{(d_i-1)d_i}{2}$ links left in the component. Then, i's payoff becomes $y\alpha^{2y+(d_i-1)d_i}$. This is a profitable deviation if and only if

$$y\alpha^{2y+(d_i-1)d_i} > d_i\alpha^{(d_i-1)d_i} \iff y\alpha^{2y} > d_i\alpha^{2d_i}.$$

Observe that the function $F(x) = x\alpha^{2x}$ is log-concave, and so it is single-peaked. Single-peakedness and $y < d_i$ implies that

$$y\alpha^{2y} > d_i\alpha^{2d_i} \implies (d_i - 1)\,\alpha^{2(d_i-1)} > d_i\alpha^{2d_i} \iff d_i > \left(1-\alpha^2\right)^{-1}.$$

y could be equal to $d_i - 1$. Therefore, there does not exist any unilateral deviation (i.e., any y) if and only if $d_i \leq (1-\alpha^2)^{-1}$ for all cliques.

Case 2 : Deviation by a pair in the same clique. Consider i and j in the same clique (so $d_i = d_j$) and a deviation in which i cuts $d_i - y$ links and now has y links, whereas j cuts $d_i - z$ links and now has z links.

We can assume i and j have not cut their links $\{i,j\}$: instead of cutting the link $\{i,j\}$ and some more links with other firms in the clique, i and j could keep the link and cut one more link with other firms in the clique. Hence, there exists a profitable deviation by i and j if and only if there exists a profitable deviation in which i and j keep their own link. So we can suppose that i and j keep their link and there are $y + z - 1 + \frac{(d_i-2)(d_i-1)}{2}$ links left in the component.

Then, i's payoff becomes $y\alpha^{2y+2z-2+(d_i-2)(d_i-1)}$ while j's payoff becomes $z\alpha^{2y+2z-2+(d_i-2)(d_i-1)}$. Without loss of generality, suppose that $y \leq z$. This is a profitable deviation if and only if

$$y\alpha^{2y+2z-2+(d_i-2)(d_i-1)} > d_i\alpha^{(d_i-1)d_i} \wedge z\alpha^{2y+2z-2+(d_i-2)(d_i-1)} > d_i\alpha^{(d_i-1)d_i} \iff$$

$$y\alpha^{2y+2z-2+(d_i-2)(d_i-1)} > d_i\alpha^{(d_i-1)d_i} \iff y\alpha^{2y+2z} > d_i\alpha^{4d_i}.$$

If $z = y$ there exists a profitable deviation in case 2 if and only if there exists y such that $y\alpha^{4y} > d_i\alpha^{4d_l}$. As in case 1, single-peakedness and $y < d_i$ implies that

$$y\alpha^{4y} > d_i\alpha^{4d_i} \implies (d_i - 1)\,\alpha^{4(d_i-1)} > d_i\alpha^{4d_i} \iff d_i > \left(1-\alpha^4\right)^{-1}.$$

Again, as in case 1, y could be equal to $d_i - 1$. In this case there does not exist any profitable deviation by a pair of firms in the same clique if and only if $d_i \leq (1-\alpha^4)^{-1}$ for all cliques.

Case 3: Deviation by a pair in different cliques. Consider i and j in different cliques who deviate jointly. If i and j do not form a link, then, this deviation must consist of two unilateral deviations in which both i and j increase their individual utilities by cutting some of their links in their own cliques. This is considered in case 1. So we can assume that case 3 consists of deviations in which the deviators i and j do form a link.

Consider the deviation in which i cuts $d_i - y$ links and adds the link with j. This leaves i with $y+1$ links. After j cuts $d_i - z$ links and adds the link with i it will have $z+1$ links. After the deviation, in the component containing i and j, there are $2y + (d_i - 1)\, d_i + 2 + 2z + (d_j - 1)\, d_j$ links.

We first show that if i or j fail to preserve all of their existing links, i.e., $y \neq d_i$ or $z \neq d_j$, this will not be a profitable deviation. Without loss of generality, suppose that $y \neq d_i$. Then, $y + 1 \leq d_i$. Therefore, i's payoff must be strictly larger than its previous payoff for this to be a profitable deviation:

$$(y+1)\, \alpha^{2y+(d_i-1)d_i+2+2z+(d_j-1)d_j} > d_i \alpha^{d_i(d_i+1)}. \tag{6}$$

From case 1 we know it suffices to consider networks in which all degrees are less than $(1-\alpha^2)^{-1}$. That is, we can assume $d_i \leq (1-\alpha^2)^{-1}$. Then $y + 1 \leq d_i \leq (1-\alpha^2)^{-1}$. By single-peakedness of $F(x) = x\alpha 2x$, we have $(y+1)\, \alpha^{2y+2} \leq d_i \alpha^{2d_i}$. Substituting this back into Equation (6) we get

$$d_i \alpha^{2d_i} \alpha^{(d_i-1)d_i+2z+(d_j-1)d_j} \geq (y+1)\, \alpha^{2y+(d_i-1)d_i+2+2z+(d_j-1)d_j} > d_i \alpha^{d_i(d_i+1)}$$

$$\implies \alpha^{2z+(d_j-1)d_j} > 1.$$

which is a contradiction.

Accordingly, we can assume that case 3 consists of deviations in which the deviating pair i and j maintain all their links in their own cliques, i.e., $y = d_i$ and $z = d_j$, and add the missing link $\{i, j\}$ between them. This is a profitable deviation if and only if

$$(d_i+1)\, \alpha^{d_i(d_i+1)+2+d_j(d_j+1)} > d_i \alpha^{d_i(d_i+1)} \wedge (d_j+1)\, \alpha^{d_j(d_j+1)+2+d_i(d_i+1)} > d_j \alpha^{d_j(d_j+1)}$$

$$\iff (d_i+1)\, \alpha^{2+d_j(d_j+1)} > d_i \wedge (d_j+1)\, \alpha^{2+d_i(d_i+1)} > d_j$$
$$\iff d_i < \left(1-\alpha^{2+d_j(d_j+1)}\right)^{-1} - 1 \wedge d_j < \left(1-\alpha^{2+d_i(d_i+1)}\right)^{-1} - 1.$$

Define $\psi : [0, \infty) \to \mathbb{R}$ as $\psi(x) := \left(1-\alpha^{2+x(x+1)}\right)^{-1} - 1$. Then, there is a profitable deviation we need to consider in case 3 if and only if there are two separate cliques, with order $d_i + 1$

and d_j+1 such that

$$d_i < \psi(d_j) \wedge d_j < \psi(d_i).$$

Note that ψ is strictly decreasing, $\psi(0) = (1-\alpha^2)^{-1} - 1 > 0$ and $lim_{x\to+\infty} = 0$. Denote by $\underline{d}$ the unique fixed point of ψ: $\psi(\underline{d}) = \underline{d}$. Observe that ψ^{-1} is well defined on the interval $(0, \psi(0)]$. Define $\phi : [0, \psi(0)] \to \mathbb{R}$ as

$$\phi(x) = \begin{cases} \psi(0) & if\, x = 0, \\ \min\{\psi(x), \psi^{-1}(x)\} & if\, x \in (0, \psi(0)], \end{cases}$$

ϕ is strictly decreasing and ϕ's unique fixed point is $\underline{d}$. Let $\overline{d} = (1-\alpha^4)^{-1}$. Next, we complete the argument that verifies the necessary and sufficient conditions given in the Theorem.

For necessity suppose first that $d_i < \underline{d}$ and $d_j < \underline{d}$. Since ψ is strictly decreasing, it follows that $\psi(d_i) > \psi(\underline{d}) = \underline{d} > d_j$ and similarly $\psi(d_j) > d_i$. Then, i and j would deviate. Second, suppose the clique that contains i is the smallest in the network ($d_i = \underline{d}_G$). Suppose that $\underline{d}_G < \underline{d}$ and $d_j < \phi(\underline{d}_G)$. Then

$$d_j < \phi(\underline{d}_G) = \min\left\{\psi(\underline{d}_G), \psi^{-1}(\underline{d}_G)\right\} \implies$$

$$d_j < \psi(\underline{d}_G) \wedge d_j < \psi^{-1}(\underline{d}_G) \implies$$

$$d_j < \psi(\underline{d}_G) \wedge \psi(d_j) > \underline{d}_G.$$

Then i and j have a profitable deviation. In sum, conditional on all degrees being less than $\overline{d}$, if there is no profitable deviation, there are no two cliques with orders strictly less than $\underline{d}+1$ or there is exactly one order strictly less than $\underline{d}+1$ but all others have orders larger than $\phi(\underline{d}_G)+1$. Now we show that, conditional on all degrees being less than $\overline{d}$, this condition is sufficient as well.

Suppose that $d_i \geq \underline{d}$ and $d_j \geq \underline{d}$. Then $d_i \geq \underline{d} \implies \psi(d_i) \leq \psi(\underline{d}) = \underline{d} \leq d_j$. Accordingly, there are no profitable deviations between cliques that have size larger than $\underline{d}+1$. Therefore, if all cliques have size larger than $\underline{d}+1$, there are no profitable deviations. Suppose that all but one have size larger than $\underline{d}+1$. Let d_i+1 be the size of the remaining clique, which has to be smallest clique: $d_i = \underline{d}_G$. Suppose that $d_j \geq \phi\left(\overline{d}_G\right)$ for all other d_j's. Then

$$d_j \geq \phi(\underline{d}_G) = \min\left\{\psi(\underline{d}_G), \psi^{-1}(\underline{d}_G)\right\} \implies$$

$$d_j \geq \psi(\underline{d}_G) \vee d_j \geq \psi^{-1}(\underline{d}_G) \implies$$

$$d_j \geq \psi\left(\underline{d}_G\right) \vee \psi\left(d_j\right) \geq \underline{d}_G.$$

Thus, there are no profitable deviations that involve the smallest clique. All other cliques have size larger than $\underline{d}+1$, so they also have no profitable deviations among each other.

Hence, we have shown in case 3 there are no profitable deviations (that have not been accounted for in other cases) if and only if either all degrees are larger than $\underline{d}$, or one clique has order less than $\underline{d}+1$, say but all the others have it larger than $\phi\left(\underline{d}_G\right)+1$. Combining all three cases concludes the proof for the necessary and sufficient condition.

For existence, L'Hopital's rule can be used to show that the gap $\overline{d}-\underline{d}$ grows unboundedly as α grows. At some point, $\overline{d}-\underline{d} \geq 2$ and so there must exist two integers in the interval $\left[\underline{d}, \overline{d}\right]$. Take two of these integers, $\lceil\underline{d}\rceil$ and $\lceil\underline{d}\rceil+1$. If $n \geq \left(\lceil\underline{d}\rceil+1\right)^2$ then

$$n-\left\lfloor\frac{n}{\lceil\underline{d}\rceil+1}\right\rfloor\left(\lceil\underline{d}\rceil+1\right) \leq \lceil\underline{d}\rceil < \lceil\underline{d}\rceil+1 \leq \left\lfloor\frac{n}{\lceil\underline{d}\rceil+1}\right\rfloor \implies$$

$$\left\lfloor\frac{n}{\lceil\underline{d}\rceil+1}\right\rfloor\left(\lceil\underline{d}\rceil+2\right)-n>0.$$

Then, the network consists of

- $n-\left\lfloor\frac{n}{\lceil\underline{d}\rceil+1}\right\rfloor\left(\lceil\underline{d}\rceil+1\right)$ many cliques of order $\lceil\underline{d}\rceil+2$ and
- $\left\lfloor\frac{n}{\lceil\underline{d}\rceil+1}\right\rfloor\left(\lceil\underline{d}\rceil+2\right)-n$ many cliques of order $\lceil\underline{d}\rceil+1$

is well-defined and bilaterally stable.

□

Necessary conditions for group stability

Proposition A 2. *A group stable network must consist of a collection of disjoint cliques, all but at most one of order d^*+1. The remaining clique must be of order at most d^*+1.*

Proof. By Proposition A1 a group stable network (if it exists) is composed of disjoint cliques. The payoff to a firm in a $(d+1)$-clique is $V(d)$. First, no clique can have order $d+1>d^*+1$ in the realized network. Otherwise, d^*+1 members would deviate by forming a (d^*+1)-clique and cutting all other links. This would be a strict improvement since d^* is the unique maximizer of $V(d)$. Second, there can be at most one clique which has order strictly less than d^*+1. Recall that $V(d)$ is single-peaked, and so it is increasing up to d^* over integers. Then, members of separate cliques each of which has less than d^* order would deviate to forming a larger clique (up to order d^*+1) to get their degree closer to d^* and improve their payoffs.

These two observations imply that there must be as many (d^*+1)-cliques as possible in the network, and the remaining part must also be a clique. $\square$

Proof of Theorem 1: group stability

Corollary of Theorem 1 in Erol and Vohra (2020)

Proof of Theorem 2: efficiency

Proof. Fix an efficient network G and select from it a component with e links and call it S. The total payoff of firms in the component S is $2e\alpha^{2e}$. Define $W:\mathbb{R}^+\to\mathbb{R}^+$ as $W(x)=2x\alpha^{2x}$. Note that W is log-concave, and so it is single-peaked. Set $k^*=\arg\max_{x\in\mathbb{N}} W(x)$ as the maximizer of W over integers. Since W is single-peaked, if $e>k^*$, by deleting one link within S, the total payoff increases. So G cannot be efficient. Hence, e must satisfy $e\leq k^*$. If $e<k^*$ and if there is an absent link within component S, then, by adding that link, the total payoff increases. So G cannot be efficient. Therefore, either $e=k^*$ or there are no absent links in the component S, i.e., S is a clique.

Suppose that S is not a clique. Then, we must have $e=k^*$. Let r be the largest integer such that $(0.5)r(r-1)+1<k^*\leq(0.5)r(r+1)$. Note, there at least $r+1$ firms in S. Otherwise, there cannot be more than $(0.5)r(r-1)$ links in S. Denote the average of payoffs of firms in S by u. Since there are at least $r+1$ firms in S and the sum of payoffs in S is $2k^*\alpha^{2k^*}$, we have $u\leq\frac{2k^*\alpha^{2k*}}{r+1}$.

Case 1: $k^*\leq(0.5)\left(r^2-1\right)$. Then, $u=\frac{2k^*\alpha^{2k*}}{r+1}\leq(r-1)\alpha^{2k^*}$. By definition of r, $(0.5)r(r-1)<k^*$, and so $u<(r-1)\alpha^{r(r-1)}=V(r-1)\leq V(d^*)$.

Case 2: $(0.5)r^2\leq k^*\leq(0.5)r(r+1)-1$. k^*+1 is an integer as well as $(0.5)r(r+1)$. Recall that k^* is the maximizer of W over integers. And so we have $W\left(k^*\right)>W\left(k^*+1\right)$. Since V is single-peaked on real numbers, $W\left(k^*\right)>W\left(k^*+1\right)$ implies that $W(x)$ is decreasing for real numbers $x\in[k^*+1,\infty]$. In particular, $W\left((0.5)r(r+1)\right)\geq W\left((0.5)r(r+2)\right)$. Then we have

$$W\left((0.5)r(r+1)\right)=r(r+1)\alpha^{r(r+1)}>W\left((0.5)r(r+2)\right)=r(r+2)\alpha^{r(r+2)}$$

$$\implies r+1>(r+2)\alpha^r \implies (r+1)\alpha^{(r-1)r}>(r+2)\alpha^{r^2}$$

$$\implies V(r-1)=(r-1)\alpha^{(r-1)r}>\frac{(r-1)(r+2)}{(r+1)}\alpha^{r^2}. \tag{7}$$

$k^*\leq(0.5)r(r+1)-1$ so $u=\frac{2k^*\alpha^{2k*}}{r+1}\leq\frac{r(r+1)-2}{r+1}\alpha^{2k^*}$. $(0.5)r^2\leq k^*$ so $\alpha^{2k^*}\leq\alpha^{r^2}$. Thus, $u\leq\frac{r(r+1)-2}{r+1}\alpha^{r^2}=\frac{(r+2)(r-1)}{r+1}\alpha^{r^2}$. By Equation (7) we have $u<V(r-1)\leq V(d^*)$.

Case 3: $k^* = (0.5)r(r+1)$. Now S is a component that has $(0.5)r(r+1)$ links but it is not a clique, so, there must be at least $r+2$ firms in S. Then, $u \leq \frac{2k^*\alpha^{2k*}}{r+2} < \frac{2k^*\alpha^{2k*}}{r+1} = r\alpha^{r(r+1)} = V(r) \leq V(d^*)$.

By combining the three cases, we see that if S is not a clique, the average payoff in S is strictly less than $V(d^*)$. Also, the average payoff in a disjoint clique is strictly less than $V(d^*)$ unless the the clique is of order d^*+1. But a group stable network achieves an average payoff of $V(d^*)$. Thus the unique efficient network consists of disjoint cliques of order d^*+1. □

Proof of Proposition 2: approximation

Proof. By single-peakedness of V, the optimal degree is given by $V(d^*-1) \leq V(d^*) \geq V(d^*+1)$. The optimal degree d^* is tied to x^* via:

$$(d-1)\alpha^{(d-1)d} \leq d\alpha^d(d+1) \iff \frac{1}{d+1} + \alpha^{2(d+1)} \leq 1 \leq \frac{1}{d} + \alpha^{2d} \iff d \leq x^* \leq d+1$$

This proves that $d^* = \lfloor x^* \rfloor$ for $\alpha \notin \tilde{A}$ and $d^* \in \{x^*-1, x^*\}$ for $\alpha \in \tilde{A}$. The rest of the proof is straightforward but cluttered algebra. We omit it. □

Proof of Proposition 3: group stability under correlated shocks

Proof. In any given network, if all shocks are bad, then, all firms default and if all shocks are good, then, all firms continue. The payoff of a firm with d links is $d\theta$ or 0 respectively. Thus, the expected payoff of each firm is $d\sigma\theta$. Then, it is clear that in a group stable (or pairwise stable) network there cannot be any missing links because that would lead to a profitable pairwise deviation. The only candidate is $(\tilde{d}+1)$-cliques which is clearly group stable. □

Proof of Theorem 3: popcorn vs. dominoes

Proof. Under popcorn, for any $\sigma \in (0,1)$, $\left(\tilde{d}+1\right)$-cliques are formed. So, the systemic risk, $1-\sigma$, can take any value between 0 and 1. Under dominoes, the necessary and sufficient condition for a complete network to be formed is, is $V\left(\tilde{d}\right) \geq V\left(\tilde{d}-1\right)$. This is because V is single-peaked. So, a complete network is formed if and only if

$$V\left(\tilde{d}\right) = \tilde{d}\alpha^{\tilde{d}\left(\tilde{d}+1\right)} \geq V\left(\tilde{d}-1\right) = \left(\tilde{d}-1\right)\alpha^{\left(\tilde{d}-1\right)\tilde{d}}$$

$$\iff \alpha \geq \left(\frac{\tilde{d}-1}{\tilde{d}}\right)^{\frac{1}{2\tilde{d}}} \iff \left(1-\left(\frac{\tilde{d}-1}{\tilde{d}}\right)^{\frac{\tilde{d}+1}{2}}\right)^{\frac{n}{\tilde{d}+1}} \geq \left(1-\alpha^{\tilde{d}\left(\tilde{d}+1\right)}\right)^{\frac{n}{\tilde{d}+1}}.$$

□

Proof of Proposition 4: expected payoffs

Remark: We use different symbols to denote parameters for normal and resilient firms in order to make the equations easier to interpret. For example, if we were to take $\zeta = \zeta'$, the formula for p_i in (2) would involve a term $\zeta^{d_i^N + d_i^M}$ instead of $\zeta^{d_i^N} \zeta'^{d_i^M}$. We find the latter easier to understand and interpret. In fact, the only parameter difference between the two types of firms needed for our results is that $\theta \neq \theta'$. All other parameters can be taken to be equal: $\gamma = \gamma'$, $\beta = \beta'$, $\alpha = \alpha'$, $\zeta = \zeta'$.

Proof. As before, a normal firm i with a good operational shock i defaults if any of its links are bad, or any of its counterparties default. This argument can be iterated along any normal path. Therefore, there are two ways that i can default. One is that any firm that can be reached from i by a normal path gets a bad operational shock, and so all normal firms along the normal path default sequentially. Second, a link of a normal firm j that can be reached from i via a normal path is bad for j, which makes j default, and so all normal firms along the normal path from j to i sequentially default. The first type of contagion is not triggered by a resilient firm with probability $(\zeta')^{d_i^M}$ and it does not get triggered by a normal firm with probability $\zeta^{d_i^N}$. The second type of contagion is not triggered with probability $\alpha^{2e_i^N + e_i^M}$. Here e_i^M is not multiplied by 2 because one firm incident to this link is a resilient firm that does not default due to link shocks. Therefore, the probability that a normal i does not default is

$$p_i = \zeta^{d_i^N} \left(\zeta'\right)^{d_i^M} \alpha^{2e_i^N + e_i^M}$$

and the expected payoff of i is $p_i d_i \theta$.

As for a resilient firm i, if i has low operational costs, i continues with all projects. This is because $\theta' > \beta' + \gamma'$ which is implied by Assumption 2. Therefore, correlations of the default probabilities of i's counterparties do not factor into i's payoff. The only risk that i defaults is that it has high operational costs. Accordingly, i's conditional expected payoff is the sum of its expected payoffs from each of its links. All of i's resilient counterparties with low operational shocks continue. Accordingly, conditional on i having low operational shocks, i has $\theta' - \alpha'\gamma' - \zeta'\beta'$ expected payoff for each of its resilient counterparties. $\alpha'\gamma'$ is the expected loss due to the fact that the link can be bad for i. $\zeta'\beta'$ is the expected loss due to the fact that its resilient counter-party can have high operational cost. As for a normal counter-party of i, say $j \in N \cap D_i$, the probability that j defaults conditional on j having low operational cost is p_j/ζ'. Accordingly, conditional on i having low operational shocks, the expected loss of i due to j's default is $p_j\beta'/\zeta'$. Summing the expected payoffs and multiplying them by the probability ζ' that i has low operational costs gives the expected payoff. □

Bilateral stability

Proposition A 3. *(Case of small core) If $d_M^* > m$, any bilaterally stable network must be a core-periphery that satisfies the following:*

1. *(Core) Each resilient firm is counterparties with all resilient firms and all normal firms.*

2. *(Periphery) Each normal firm is counterparties with all resilient firms and some normal firms. Excluding resilient firms and their incident links, normal firms form disjoint cliques among themselves.*

(Case of large core) If $d_M^* \leq m$, any bilaterally stable network must be a core-periphery that satisfies the following:

1. *(Core) Each resilient firm is counterparties with all resilient firms and some normal firms.*
2. *(Periphery) Each normal firm has d_M^* resilient counterparties and no normal counterparties.*

Proof. Fix a network. A resilient firm i always wants to link with all other firms. Accordingly, in any bilaterally stable network, all pairs of resilient firms have links. Suppose that there exists a normal firm j that has a link with another normal firm k, but not with i. Then j would sever its link with k, and i and j would form a link. This strictly improves i's payoff. As for j, p_j is reduced by $\alpha^2\zeta$ by cutting the link with k and increased by $\zeta'\alpha$ by adding the link with i. Since $\zeta' > \zeta\alpha$, p_j decreases strictly. j's degree does not change, and so j's payoff strictly improves. Therefore, in any bilaterally stable network, any normal firm with a normal counter-party must be counterparties with all resilient firms. So normal firms either have only resilient counterparties, or have resilient firms as counterparties and potentially other normal counterparties.

Take a normal firm j that has a resilient counter-party i and a normal counter-party k. If k is not counterparties with i, k would sever its link with j and form a link with i. This would improve the payoffs of both i and k. So i and k must already be counterparties. That is, all normal counterparties of i must be counterparties with all resilient firms.

Now focus on the periphery. The proof of Proposition 1 applies, and so the periphery must be in cliques. Combine this with previous argument to see that a normal i firm is either in a clique in the periphery where the entire clique is adjacent to all of the resilient firms, or i does not have any normal counterparties and all of its counterparties are resilient firms.

Consider the case of $m < d_M^*$. Take a normal firm i. Since $d\left(\zeta'\alpha\right)^d$ is single-peaked (due to log-concavity) and $m < d_M^*$, i would want to form links with more resilient firms, and resilient firms would be happy to correspond. So in any bilaterally stable network, all normal firms must be counterparties of with all resilient firms. The periphery must be in disjoint cliques among themselves.

Consider the case of $m > d_M^*$. Normal firms would like to have d_M^* many counterparties and there is sufficiently many resilient firms to form links with. Then all normal firms have d_M^* many resilient counterparties and no normal counterparties.

□

Proof of Theorem 4: group stability

Proof. Consider the case $m < d_M^*$. In a group stable network, for a normal firm i, we have $d_i^M = m$ since all resilient firms can be immediately reached from i. Moreover, all normal firms j that can be reached from i via normal firms are also counterparties with all resilient firms, so that $e_i^M = md_i^N$. Therefore, $p_i = \zeta^{d_i^N}\left(\zeta'\right)^m \alpha^{2e_i^N + md_i^N} = \left(\alpha^m\zeta\right)^{d_i^N} \alpha^{2e_i^N} \times \left(\zeta'\right)^m$. Accordingly, i's expected payoff is

$$d_i\left(\alpha^m\zeta\right)^{d_i^N} \alpha^{2e_i^N} \left(\zeta'\right)^m \theta.$$

Note that e_i^N is the number of links and d_i^N is the number of firms in the component of i in the subnetwork of normal firms. Accordingly, one can ignore all resilient firms, focus on only normal firms, and study group stable networks with the payoff function

$$U\left(d,e\right) = \left(d+m\right)\left(\alpha^m\zeta\right)^{d+1}\alpha^{2e}.$$

This payoff function is decreasing in e and so the proof of Theorem 1 can be replicated to show the normal firms must be formed into disjoint cliques of order $d_N^* + 1$ and be connected to all resilient firms, and that there is no profitable deviation from this network.

Consider the case $m > d_M^*$. Now any bilaterally stable network is group stable. Normal firms have the maximum payoff they can achieve in any given network because they are connected to the least risky counterparties at their individual rational level. Normal firms can not be convinced to deviate. Clearly, the resilient firms do not want to alter their links among themselves. □

Proof of Theorem 5: efficiency

Proof. In an efficient network it is clear that all links between resilient firms should be formed. We now argue that each normal firm should have a link with each resilient firm. Choose any component S in the subnetwork induced by the set of normal firms where at least one normal firm is not linked to at least one resilient firm. Let $V(S)$ be the set of firms in the components S. Below we show that the total expected payoff of the firms in S is bounded above by a constant v^* that is independent of S and θ'.

Regardless of the rest of the network, the gain from adding an absent link between a resilient firm and a normal firm is larger than $\theta' - \alpha'\gamma' > (1-\alpha')\theta'$ which for θ' sufficiently large exceeds v^*. So regardless of how much of a negative externality this link can impose on the rest of the network, it is efficient to form this link. Therefore, the resilient firms have links with all normal firms.

Now we argue that no two normal firms should be adjacent. Suppose, for a contradiction, there is a link between i and j. Let $1-p_i$ be the default probability of i. Removing this link can reduce i's payoff by at most $\theta d_i p_i - \theta(d_i-1)p_i/\alpha^2 < \theta p_i - \theta p_i m(1/\alpha^2-1)$. The payoffs of all other normal firms weakly increase, so the total loss to normal firms is at most $\theta(p_i+p_j)(1-m(1/\alpha^2-1))$. The gain to resilient firms is at least $m\gamma'(1/\alpha^2-1)(p_i+p_j)$ which is larger than the loss if γ' is sufficiently large. So there can not be any links between normal firms.

We now establish the existence of v^*. We use equation (2). For all $i,j \in V(S)$ we have $d_i^N = d_j^N = |V(S)|$, $d_i^M = d_j^M = d^M$, $e_i^N = e_j^N = e^N$ and $e_i^M = e_j^M = e^M$. If d_i is the degree of each $i \in V(S)$, the sum of expected payoffs of the firms in $V(S)$ is

$$\sum_{i\in V(S)} d_i \theta \zeta^{|V(S)|} \zeta'^{d^M} \alpha^{2e^N+e^M} = 2e^N \theta \zeta^{|V(S)|} \zeta'^{d^M} \alpha^{2e^N+e^M}.$$

This is bounded above by $(2e^N+e^M)\alpha^{2e^N+e^M}$. This is an expression of the form $x\alpha^x$ which attains a unique maximum value we call v^*. It is clearly independent of S and θ'.

□

Mapping the model to the microfoundation

Proposition A 4. *Suppose that $K_iP_i > L_i\left(\overline{B}_i - \underline{C}_i - P_iR_i\right) > 0$ and $\underline{B}_i > \overline{C}_i$ for all i. Firm i exists if it has high operational costs. Otherwise, the best response of i with low operational costs is to either keep all projects or to drop all projects. If i drops all projects, its payoff is 0. If i keeps all projects as a best response, its payoff is $d_i\theta_i - f_i\beta_i - b_i\gamma_i$.*

Proof. Firm i can have at most L_i/K_i projects, each of which can generate at most $\overline{B}_i - \underline{C}_i - P_iR_i$. Thus $K_iP_i > L_i\left(\overline{B}_i - \underline{C}_i - P_iR_i\right)$ implies that i defaults if it has a high operational cost. Now focus on i that has low operational cost.

Notation: For all i and for all $j \in D_i$, set $a_{ij} = 1$ if firm i keeps project $\{i, j\}$, and $a_{ij} = 0$ otherwise. Denote by $a_i = (a_{ij})_{j \in D_i}$ the strategy of i. Let $a = (a_i)_{i \le n}$ be the action profile played by all firms. Denote by $a_{-i} = (a_j)_{j \ne i}$ the action profile played by firms other than i. Denote by $\mathbf{1}$ the vector of 1's and $\mathbf{0}$ the vector of 0's.

Ex-post payoffs: For $X \in \{0, 1\}$ and $Y \in \left\{\overline{C}, \underline{C}\right\}$, let

$$D^i_{X,Y}(a) = \left|\left\{ j \in D_i \right| a_{ji} = X, C^{ij} = Y \right\}\right|.$$

This is the number of counterparties j of i such that the counter-party j is playing X for its project $\{i, j\}$ with i, whereas i's management cost for the project $\{i, j\}$ with j is Y. Then

$$\left(\left|D^i_{1,\underline{C}}\right| + \left|D^i_{1,\overline{C}}\right|\right)\overline{B} + \left(\left|D^i_{0,\underline{C}}\right| + \left|D^i_{0,\overline{C}}\right|\right)\underline{B}$$

is the revenue of firm i. Also, $d_i PR$ must be repaid back to lenders. Let

$$\Pi_i(a) = \left(\left|D^i_{1,\underline{C}}\right| + \left|D^i_{1,\overline{C}}\right|\right)\overline{B} + \left(\left|D^i_{0,\underline{C}}\right| + \left|D^i_{0,\overline{C}}\right|\right)\underline{B} - d_i PR.$$

Since firms are protected by limited liability, i has profit $(\Pi_i(a))^+$. On top of the profit, i incurs, the effort cost of managing projects (operational cost), is

$$C_i(a) = \left(\left|D^i_{1,\overline{C}}(a)\right| + \left|D^i_{0,\overline{C}}(a)\right|\right)\overline{C} + \left(\left|D^i_{0,\underline{C}}\right| + \left|D^i_{1,\underline{C}}\right|\right)\underline{C}.$$

Accordingly, i's payoff is

$$U_i(a) = (\Pi_i(a))^+ - C_i(a).$$

$U_i(a)$ can fall below zero because the management costs of projects, $\underline{C}$ and $\overline{C}$, are utility costs of effort. If i continues with projects despite having very small or zero profit, U_i can be negative.

Best responses: Define

$$V_i(a) = \Pi_i(a) - C_i(a).$$

By $\underline{B}_i > \overline{C}_i$, for any a_{-i}, $V_i(a'_i, a_{-i})$ is increasing in a'_i. In particular, for any a_{-i}, $\arg\max_{a'_i} V_i(a'_i, a_{-i}) = \mathbf{1}$ is the unique maximizer.

Case 1: If $V_i(\mathbf{1}, a_{-i}) > 0$, then $\Pi_i(\mathbf{1}, a_{-i}) > C_i(\mathbf{1}, a_{-i}) > 0$. So $U_i(\mathbf{1}, a_{-i}) = V_i(\mathbf{1}, a_{-i}) > 0$. Consider any $a'_i < \mathbf{1}$. If $\Pi_i(a) \le 0$, then $U_i(a'_i, a_{-i}) \le 0 < U_i(\mathbf{1}, a_{-i})$. If $\Pi_i(a) > 0$, then $U_i(a'_i, a_{-i}) = V_i(a'_i, a_{-i}) < V_i(\mathbf{1}, a_{-i}) = U_i(\mathbf{1}, a_{-i})$. Thus $a'_i = \mathbf{1}$ is the unique best response.

Case 2: If $V_i(\mathbf{1}, a_{-i}) < 0$, then $V_i(a'_i, a_{-i}) < 0$ for all a'_i. Then $\Pi_i(a'_i, a_{-i}) < C_i(a'_i, a_{-i})$

for all a_i'. Also, for all $a_i' \neq \mathbf{0}$, $0 < C_i(a_i', a_{-i})$. Then combining both, for all $a_i' \neq \mathbf{0}$, $(\Pi_i(a_i', a_{-i}))^+ < C_i(a_i', a_{-i})$. So $U_i(a_i', a_{-i}) < 0$ for all $a_i' \neq \mathbf{0}$. Then the unique best response is $a_i' = \mathbf{0}$ since it yields $U_i(\mathbf{0}, a_{-i}) = 0$.

Case 3: If $V_i(\mathbf{1}, a_{-i}) = 0$, then $V_i(a_i', a_{-i}) < 0$ for all $a_i' \neq \mathbf{1}$. Similar to Case 2, we get $U_i(a_i', a_{-i}) < 0$ for all $a_i' \neq \mathbf{0}, \mathbf{1}$. For $a_i' = \mathbf{0}$, $U_i(\mathbf{0}, a_{-i}) = 0$. For $a_i' = \mathbf{1}$, by $V_i(\mathbf{1}, a_{-i}) = 0$, we have $\Pi_i(\mathbf{1}, a_{-i}) = C_i(\mathbf{1}, a_{-i}) > 0$. So $U_i(\mathbf{1}, a_{-i}) = V_i(\mathbf{1}, a_{-i}) = 0$. Then both $a_i' \in \{\mathbf{0}, \mathbf{1}\}$ are the best responses.

This establishes that the best response is either to keep all projects ($a_i' = \mathbf{1}$ when $V_i(\mathbf{1}, a_{-i}) \geq 0$) or to drop all projects ($a_i' = \mathbf{0}$ when $V_i(\mathbf{1}, a_{-i}) \leq 0$). Observe that

$$
\begin{aligned}
V_i(\mathbf{1}, a_{-i}) = \left(\left|D^i_{1,\underline{C}}(\mathbf{1}, a_{-i})\right| + \left|D^i_{1,\overline{C}}(\mathbf{1}, a_{-i})\right|\right)\overline{B} + \left(\left|D^i_{0,\underline{C}}(\mathbf{1}, a_{-i})\right| + \left|D^i_{0,\overline{C}}(\mathbf{1}, a_{-i})\right|\right)\underline{B} - d_i PR \\
- \left(\left(\left|D^i_{1,\overline{C}}(\mathbf{1}, a_{-i})\right| + \left|D^i_{0,\overline{C}}(\mathbf{1}, a_{-i})\right|\right)\overline{C} + \left(\left|D^i_{0,\underline{C}}(\mathbf{1}, a_{-i})\right| + \left|D^i_{1,\underline{C}}(\mathbf{1}, a_{-i})\right|\right)\underline{C}\right) \\
= d\left(\overline{B} - \underline{C} - P\right) - \left(\left|D^i_{0,\overline{C}}(\mathbf{1}, a_{-i})\right| + \left|D^i_{0,\underline{C}}(\mathbf{1}, a_{-i})\right|\right)\left(\overline{B} - \underline{B}\right) \\
- \left(\left|D^i_{1,\underline{C}}(\mathbf{1}, a_{-i})\right| + \left|D^i_{0,\underline{C}}(\mathbf{1}, a_{-i})\right|\right)\left(\overline{C} - \underline{C}\right) \\
= d_i\theta - f_i\beta - b_i\gamma.
\end{aligned}
$$

□

Proposition A 5. *Suppose that* $K_iP_i > L_i\left(\overline{B}_i - \underline{C}_i - P_iR_i\right) > 0$ *for all* i. *Call* i *a high-rate firm if*

$$
0 < L_i\left(\frac{\overline{B}_i - \underline{C}_i}{P_i} - R_i\right) < \min\left\{\left(\overline{B}_i - \underline{B}_i\right), \left(\overline{C}_i - \underline{C}_i\right)\right\}.
$$

and a low-rate firm if

$$
\underline{B}_i - \overline{C}_i > P_iR_i
$$

The following hold:

- *Consider* n *identical high-rate firms. Also take* $\zeta = 1$. *Then microfoundation is equivalent to the model of periphery in Section 2.*
- *Consider* n *identical high-rate firms and* m *identical low-rate firms. Also* $\zeta_i > \zeta_j\alpha_j$ *for all low-rate* i *and all high rate* j. *Then, the microfoundation is equivalent to the model of core-periphery in Section 3.*

Proof. $K_iP_i > L_i\left(\overline{B}_i - \underline{C}_i - P_iR_i\right)$ implies that i defaults if it has bad node shock (high operational cost).

$\underline{B}_i - \overline{C}_i - P_iR_i > 0$ implies that $\theta_i > \beta_i + \gamma_i$. So, low rate firms in the micro-foundation are resilient firms in the reduced form model.

$\frac{\overline{B}_i - \underline{C}_i}{P_i} - R_i > 0$ implies $\theta_i > 0$. On the other hand,

$$L_i \left(\frac{\overline{B}_i - \underline{C}_i}{P_i} - R_i \right) < \min \left\{ \left(\overline{B}_i - \underline{B}_i \right), \left(\overline{C}_i - \underline{C}_i \right) \right\}$$

implies $\frac{L_i}{P_i} \theta_i < \min\{\beta_i, \gamma_i\}$, which implies $\tilde{d}_i \theta_i < \min\{\beta_i, \gamma_i\}$. Thus, high rate firms are normal firms in the reduced form model.

When all firms are high rate and $\zeta_i = 1$, the microfoundation boils down to n normal firms without node shocks, which is the model in Section 2. $\zeta_i > \zeta_j \alpha_j$ for all low rate i and high rate j corresponds to Assumption 3. Then, the model with n high rate and m low rate firms corresponds to the model in Section 3.

□

Proof of Theorem 6: existence of core-periphery with ex-ante identical agents

Proof. Under interest rates $\zeta^{-}1$ and $\zeta^{-1} (\alpha\zeta)^{-\left\lfloor \frac{1}{1-\alpha\zeta} \right\rfloor}$, Assumption M1-3 imply the conditions of Proposition A5. Then, by Theorem 4, the network formed is group stable.

In particular, $d_M^* = \arg\max \ d (\alpha\zeta)^d = \left\lfloor \frac{1}{1-\alpha\zeta} \right\rfloor < m$. Since $d_M^* < m$, $d_N^* = 0$. (The network is given by the case of the large core in Theorem 4.) Then, the default probability of a firm in the periphery is $1 - \zeta (\alpha\zeta)^{\left\lfloor \frac{1}{1-\alpha\zeta} \right\rfloor}$. The default probability of a firm in the core is $1 - \zeta$. Then, the interest rate that makes lenders of periphery firms indifferent is $\zeta^{-1} (\alpha\zeta)^{-\left\lfloor \frac{1}{1-\alpha\zeta} \right\rfloor} = R$. The interest rate that makes lenders of the core firms indifferent is $\zeta^{-1} = R'$. □

Analysis of other core-periphery equilibria with ex-ante identical agents

Let $\overline{R} = \frac{\overline{B} - \underline{C}}{P}$ and $\underline{R} = \overline{R} - \frac{\min\left\{\overline{B} - \underline{B}, \overline{C} - \underline{C}\right\}}{L}$.

Definition 1. *An equilibrium is called a* ***core-periphery equilibrium*** *if one group of firms (called the core) faces interest rate ζ^{-1} and the other group (the periphery) faces an interest rate between $\underline{R}$ and $\overline{R}$.*

For given R and m, a core-periphery equilibrium is called an ***(R, m)-equilibrium*** *if the interest rate for the periphery is R and the size of the core is m.*

So as not to be distracted by extraneous details caused by integrality requirements we focus on values of α and ζ for which these requirements are moot. For a given m, denote by $x_M^*(m, \alpha, \zeta) \in \mathbb{R}$ the solution to $\max_{x_M \in [0,m]} x_M (\alpha\zeta)^{x_M}$ and denote by $x_N^*(m, \alpha, \zeta) \in \mathbb{R}$ the solution to $\max_{x_N \geq 0} (m + x_N) \zeta^m \alpha^{(m+x_N)(x_N+1)}$. Here x_M^* and x_N^* are 'smooth' counterparts of d_M^* and d_N^*. Let $\Omega = \{(m, \alpha, \zeta) | x_M^*(m, \alpha, \zeta), x_N^*(m, \alpha, \zeta) \in \mathbb{Z}\}$. By focusing on values in Ω we need not worry about discrete jumps.

Theorem A 1. *(Characterization) Suppose that Assumption M1 and 2 hold. Focus on values in* Ω. *Fix* α, ζ *and drop them from notation for simplicity.*

For any (R,m)*-equilibrium with* $R \in [\underline{R}, \overline{R}]$, *the periphery has* $x_M^*(m)$ *links with the core and* $x_N^*(m)$ *links with the periphery. The interest rate* R *satisfies*

$$R^{-1} = \zeta \left(\zeta \alpha^{x_N^*(m)+1}\right)^{x_M^*(m)+x_N^*(m)}.$$

If $R > \zeta^{-1}\boldsymbol{e}$, *there is no* (R,m)*-equilibrium for any* m.

Denote by $\underline{m}$ *the positive solution to* $1 + m\ln(\alpha\zeta) + m^2\ln(\alpha) = 0$ *and* $\overline{m}$ *the positive solution to* $1 + m\ln(\alpha\zeta) = 0$.

If $R = \zeta^{-1}\boldsymbol{e}$, *there is an* (R,m)*-equilibrium if and only if* $m > \overline{m}$. *For an* (R,m) *equilibrium,* $x_M^*(m) = \overline{m}$ *and* $x_N^*(m) = 0$. (R,m) *equilibria with higher* m *are more efficient provided that* $\zeta > (\boldsymbol{e}-1)^{-1}$.

If $\zeta^{-1}\mathbf{e}\,(\alpha\zeta)^{\sqrt{1-\frac{4\ln(\alpha)}{(\ln(\alpha\zeta))^2}}} \leq R < \zeta^{-1}\boldsymbol{e}$, *there exists* $m_1(R)$, $m_2(R)$ *with* $m_1(R) \leq \underline{m} \leq m_2(R) \leq \overline{m}$ *such that there exists an* (R,m)*-equilibrium if and only if* $m \in \{m_1(R), m_2(R)\}$. *For these, we have* $x_M^*(m_1(R)) = m_1(R)$, $x_N^*(m_1(R)) > 0$, *and* $x_M^*(m_2(R)) = m_2(R)$, $x_N^*(m_2(R)) = 0$. $(R, m_2(R))$*-equilibrium is more efficient than* $(R, m_1(R))$*-equilibrium.*[21] m_1 *is increasing and* m_2 *is decreasing. Under large enough* k, $(m_i^{-1}(m), m)$ *equilibrium is more efficient for larger* m.

If $R < \zeta^{-1}\mathbf{e}\,(\alpha\zeta)^{\sqrt{1-\frac{4\ln(\alpha)}{(\ln(\alpha\zeta))^2}}}$, *there is no* (R,m)*-equilibrium for any* m.

(R,m)-equilibria under $R = \zeta^{-1}\mathbf{e}$ necessitates $m > \overline{m}$. Accordingly we call these **large-core equilibria**.[22] (R,m)-equilibria under $R < \zeta^{-1}\mathbf{e}$ have core size $m_1(R)$ or $m_2(R)$. Recall that $m_1(R) \leq \underline{m} \leq m_2(R) \leq \overline{m}$. Accordingly, we call $(R, m_2(R))$-equilibria the **medium-core equilibria** and $(R, m_1(R))$-equilibria the **small-core equilibria**.

Proof. First, note that (R,m)-equilibria constitute a partition of core-periphery equilibria Notice that each core-periphery equilibrium is an (R,m)-equilibrium for the corresponding R and m.

Assumption M1 and 2, and $R \in [\underline{R}, \overline{R}]$ imply the conditions of Proposition A5. Then, by Proposition A5, (R,m)-equilibrium networks in are characterized by networks in Theorem

[21] $m_1(R) = m_2(R)$ only at $R = \zeta^{-1}\mathbf{e}\,(\alpha\zeta)^{\sqrt{1-\frac{4\ln(\alpha)}{(\ln(\alpha\zeta))^2}}}$

[22] Note that for values in Ω, we have $\zeta^{-1}(\alpha\zeta)^{-\left\lfloor\frac{1}{1-\alpha\zeta}\right\rfloor} = \zeta^{-1}\mathbf{e}$ so that Theorem A1 implies Theorem 6 on Ω.

4 which involve interest rate R for the periphery firms. In particular, periphery firms have $x^*_M(m)$ core counterparties, $x^*_N(m)$ periphery counterparties, and the periphery is organized into cliques of order $x^*_N(m)+1$. This makes the continuation probability of a periphery firm $\zeta\left(\zeta\alpha^{x^*_N(m)+1}\right)^{x^*_M(m)+x^*_N(m)}$ which is R^{-1}.

Some algebra shows that following hold. If $\overline{m}\le m$ (large-core case), then $x^*_M(m)=\overline{m}$ and $x^*_N(m)=0$. In this case, $R^{-1}=\zeta(\alpha\zeta)^{\overline{m}}=\zeta\mathbf{e}^{-1}$. If $\underline{m}\le m<\overline{m}$ (medium-core case), then $x^*_M(m)=m$ and $x^*_N(m)=0$. In this case, $R^{-1}=\zeta(\zeta\alpha)^m>\zeta(\zeta\alpha)^{\overline{m}}=\zeta\mathbf{e}^{-1}$. If $m<\underline{m}$ (small-core case), then $x^*_M(m)=m$ and $x^*_N(m)>0$. In this case, $R^{-1}=\zeta(\zeta\alpha^{x^*_N(m)+1})^{m+x^*_N(m)}=\zeta\mathbf{e}^{-1}\alpha^{-(x^*_N(m)+m)^2}>\zeta\mathbf{e}^{-1}$. (The last equality follows from the definition of $x^*_N(m)$ and the FOC therein.)

Using these we immediately see that if $R>\zeta^{-1}\mathbf{e}$, there is no (R,m)-equilibrium for any m. If $R=\zeta^{-1}\mathbf{e}$, there is an (R,m)-equilibrium if and only if $m>\overline{m}$. In this case the sum of payoffs of all firms is

$$m(m-1)\zeta\left(\theta_c-(1-\alpha)\gamma-(1-\zeta)\beta\right)+\overline{m}(k-m)\left(\zeta\left(\theta_c-(1-\alpha)\gamma-(1-R^{-1})\beta\right)+R^{-1}\theta_p\right)$$

where $\theta_c=\overline{B}-\underline{C}-P\zeta^{-1}1$ and $\theta_p=\overline{B}-\underline{C}-PR$. Using $\theta_c>\beta+\gamma$, $\theta_p<\beta$, $R=\zeta^{-1}\mathbf{e}$, $m>\overline{m}$, $m\ge 1$, some algebra yields that the sum of payoff is increasing in m if $\zeta>(e-1)^{-1}$

For $\zeta^{-1}\mathbf{e}\,(\alpha\zeta)^{\sqrt{1-\frac{4\ln(\alpha)}{(\ln(\alpha\zeta))^2}}}<R<\zeta^{-1}\mathbf{e}$, there are two solutions: one from the medium-core case and one from the small-core case. The solution from the medium-core case is $m_2(R)$ that solves $R^{-1}=\zeta(\alpha\zeta)^{m_2(R)}$, i.e., $m_2(R)=\frac{\ln(R\zeta)}{-\ln(\alpha\zeta)}$. The solution from the small core case is $m_1(R)$ that solves $R^{-1}=\zeta(\zeta\alpha^{x^*_N(m_1(R))+1})^{m_1(R)+x^*_N(m_1(R))}$.

$$\zeta(\alpha\zeta)^{m_2(R)}=\zeta(\zeta\alpha^{x^*_N(m_1(R))+1})^{m_1(R)+x^*_N(m_1(R))}$$

implies that $m_2(R)\ge m_1(R)+x^*_N(m_1(R))$. Under m_1, the payoff of a periphery firm is $u_{p,1}=(m_1+x^*_N(m_1))R^{-1}\theta_p$. Under m_2, it is $u_{p,2}=m_2R^{-1}\theta_p>u_{p,1}$. Under m_i, the payoff of a core firm is

$$u_{c,i}=(m_i-1)\zeta\left(\theta_c-(1-\alpha)\gamma-(1-\zeta)\beta\right)+(k-m_i)\zeta\left(\theta_c-(1-\alpha)\gamma-(1-R^{-1})\beta\right)$$

which is partially increasing in m_i. So the core firms are better off under m_2. Clearly the core firms are better off than the periphery firms. Therefore, $u_{c,2}>u_{p,2}$, $u_{c,1}>u_{p,1}$, $u_{c,2}>u_{c,1}$, $u_{p,2}>u_{p,1}$. Then by $m_2>m_1$, we find that m_2 is more efficient than m_1.

It is easy to show that m_1 is increasing and m_2 is decreasing in R.

As for sum of payoffs w.r.t. m, for large k and $m=m_2(R)$,

$$
\begin{aligned}
u_{p,2} &= m\zeta(\zeta\alpha)^m\theta_p \\
\frac{d\frac{\sum u_{p,2}}{k}}{dm} &\approx \tfrac{d}{dm} m\zeta(\zeta\alpha)^m\theta_p > 0 \text{ (because } m < \overline{m}\text{)} \\
u_{c,2} &= (m-1)\zeta\left(\theta_c - (1-\alpha)\gamma - (1-\zeta)\beta\right) + (k-m)\zeta\left(\theta_c - (1-\alpha)\gamma - (1-\zeta(\zeta\alpha)^m)\beta\right) \\
\frac{d\frac{\sum u_c}{k}}{dm} &\approx \tfrac{d}{dm} m\zeta\left(\theta_c - (1-\alpha)\gamma - (1-\zeta(\zeta\alpha)^m)\beta\right) = \zeta\left(\theta_c - (1-\alpha)\gamma - \beta + \zeta\beta\tfrac{d}{dm} m(\zeta\alpha)^m\right) > 0
\end{aligned}
$$

For large k and $m = m_1(R)$,

$$
\begin{aligned}
u_{p,1} &= d_m\zeta e^{-1}\alpha^{-d_m^2} \\
&\quad \text{(where } d_m = m + x_N^*(m)\text{, which is increasing in } m\text{)} \\
\frac{d\frac{\sum u_{p,1}}{k}}{dm} &\approx \tfrac{d}{dm} u_{p,1} > 0 \\
u_{c,1} &= (m-1)\zeta\left(\theta_c - (1-\alpha)\gamma - (1-\zeta)\beta\right) \\
&\quad + (k-m)\zeta\left(\theta_c - (1-\alpha)\gamma - (1-\zeta e^{-1}\alpha^{-d_m^2})\beta\right) \\
\frac{d\frac{\sum u_{c,1}}{k}}{dm} &\approx \tfrac{d}{dm}\partial m\zeta\left(\theta_c - (1-\alpha)\gamma - (1-\zeta e^{-1}\alpha^{-d_m^2})\beta\right) > 0
\end{aligned}
$$

At $R = \zeta^{-1}\mathbf{e}\,(\alpha\zeta)^{\sqrt{1-\frac{4\ln(\alpha)}{(\ln(\alpha\zeta))^2}}}$, the unconstraint $x_N^*(m)$ is equal to 0. This is the point at which $m_1(R) = m_2(R)$. Below this R there is no equilibrium. □

Appendix B Supplementary Material

B.1 Other measures of systemic risk

As stated in Section 2.3, our insights are valid under other measures of systemic risk as well. Here we formalize this. Given α, the number of defaults is equal to $(d^*+1)s$ with probability

$$\binom{\frac{n}{d^*+1}}{s}\left(1-\alpha^{d^*(d^*+1)}\right)^s\left(\alpha^{d^*(d^*+1)}\right)^{\frac{n}{d^*+1}-s}. \tag{8}$$

There is no first order stochastic dominance order among these distributions indexed by α. However, the distributions with larger α's second order stochastically dominate those with smaller α's. Approximately, $s\left(-2\log[\alpha]\right)$ firms default out of n, with probability $\mathbb{F}\left[s,\left\lfloor n\sqrt{-2\log[\alpha]}\right\rfloor,1-\boldsymbol{e}^{-0.5}\right]$, where $\mathbb{F}$ is the binomial pdf.

The mean and variance of the number of defaults are given by

$$\frac{\mu_{defaults}}{n}=\left(1-\alpha^{d^*(d^*+1)}\right)\approx\left(1-\boldsymbol{e}^{-0.5}\right),$$

$$\frac{\sigma^2_{defaults}}{n}=(d^*+1)\left(1-\alpha^{d^*(d^*+1)}\right)\left(\alpha^{d^*(d^*+1)}\right)\approx\left(-2\log[\alpha]\right)^{-0.5}\left(1-\boldsymbol{e}^{-0.5}\right)\boldsymbol{e}^{-0.5}.$$

Except for the discreteness, the mean is arguably constant but the variance gets larger in α. Increasing variance is due to the fact that firms in components have correlated risk and the components are getting larger in α. Figure 4 shows how the mean and variance vary with α.

There are other measures of systemic risk besides the probability that all firms default. Figure 5 shows the probability that at least 90% of firms default and the probability that at least 50% of firms default. These measures of systemic risk also increase in α, meaning that our insight regarding the volatility paradox is not specific to the notion of systemic risk defined as the probability of all firms defaulting.

Despite the constant mean and increasing volatility in the number of defaults, mean welfare increases in α. The sum off payoffs is $nd^*\alpha^{d^*(d^*+1)}$ which is clearly increasing in α by definition of d^* being the maximizer of $V(d)$. The distribution of welfare can also be pinned down. Realized ex-post welfare is $(n-(d^*+1)k)\,d^*\theta$ with probability given in Equation (8). Hence welfare has mean and variance given by

$$\frac{\mu_{welfare}}{n}=d^*\alpha^{d^*(d^*+1)}\theta\approx\boldsymbol{e}^{-0.5}\left(-2log[\alpha]\right)^{-1}\theta,$$

$$\frac{\sigma^2_{welfare}}{n} = (d^*+1)(d^*)^2\theta^2\left(1-\alpha^{d^*(d^*+1)}\right)\left(\alpha^{d^*(d^*+1)}\right) \approx (-2log[\alpha])^{-1.5}\left(1-\boldsymbol{e}^{-0.5}\right)\boldsymbol{e}^{-0.5}\theta^2.$$

Both means and variance are increasing in α. These are plotted in Figure 6. The volatility paradox holds for welfare as well: welfare has higher mean and higher variance in fundamentally safer economies.

B.2 Robustness: Stability and volatility paradox

We relax Assumption 1 and check for the robustness of volatility paradox. We start with stability of the clique structure. We consider networks that consist of disjoint cliques of order d^*, which is defined as the degree that maximizes the average payoff of firms in the clique.

Unfortunately, we are unable to do this for any set of parameters. We maintain the assumption $\gamma > \tilde{d}\theta$. That is, one bad shock causes the firm to default, but we relax $\beta > \tilde{d}\theta$, so that a nontrivial number of counterparty defaults can cause a firm to default. For example, for $\theta = 1$ and $\beta = 2$, a firm with good shocks defaults only if more than half its counterparties default. Accordingly, in a disjoint clique, if more than half the firms suffer bad shocks, all firms default.

In this case, the payoff of a firm in a $(d+1)$-clique is not $d\alpha^{d(d+1)}\theta$ anymore. Denote by $\mathbb{P}_{bin}$ and $\mathbb{F}_{bin}$the binomial PDF and CDF. The expected payoff of a firm in a $(d+1)$-clique becomes

$$V(d) = \alpha^d \times \sum_{f=0}^{\left\lfloor \frac{d\theta}{\beta}\right\rfloor} \mathbb{P}_{bin}\left[f;d,1-\alpha^d\right](d\theta-\beta f)$$

$$= d\alpha^d \times \left\{\theta\mathbb{F}_{bin}\left[\left\lfloor\frac{d\theta}{\beta}\right\rfloor;d,1-\alpha^d\right] - \beta\left(1-\alpha^d\right)\mathbb{F}_{bin}\left[\left\lfloor\frac{d\theta}{\beta}\right\rfloor-1;d-1,1-\alpha^d\right]\right\}. \quad (9)$$

Proposition A 6. *Suppose that $\alpha^2 + \frac{\theta}{\beta} < 1$. Then, a network that consists of cliques of order d^*+1 is bilaterally stable.*

The proof is given at the end of the section.

Next we consider the volatility paradox. Absent a closed-form expression for d^* we resort to simulations for comparative statics. Consider $\theta = 1$ and $\beta = 1.5$. d^* increases in α. Figure 11 plots the corresponding systemic risk. Systemic risk increases as the probability of good shock α increases, even under mild network externalities. Therefore, the volatility paradox is not just a consequence of Assumption 1.

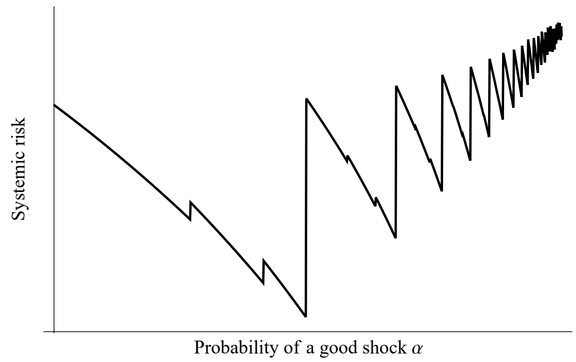

Figure 11: Peripheral volatility paradox persists under weaker contagion: $\theta = 1$, $\beta = 1.5$

In order to reinforce our argument, consider $\beta < 1$. Then, there is no contagion and the only risk that a firm faces is the immediate shocks to its links. A firm's payoff is $d\alpha^d\theta$. Note that $\arg\max d\alpha^d = \left\lfloor \frac{1}{1-\alpha} \right\rfloor$, which is approximately $(-\ln(\alpha))^{-1}$ for large α. The default risk of a firm is approximately $1 - \mathbf{e}^{-1}$. Systemic risk is $(1 - \mathbf{e}^{-1})^n$, which is constant.

These illustrate that in our model the volatility paradox arises due to network externalities. If there is no contagion, there is no volatility paradox.

Similarly for the core-periphery, we argue that central-volatility paradox is a consequence of contagion, not parametric assumptions. We focus on the large core case (m is sufficiently large) in order to single out the central volatility paradox. Consider a network that consists of singletons in the periphery, such that each peripheral firm has links with d many core firms out of m. The expected payoff of a peripheral firm is

$$V(d) = \alpha^d \zeta \times \sum_{f=0}^{\left\lfloor \frac{d\theta}{\beta} \right\rfloor} \mathbb{P}_{bin}[f; d, 1-\zeta](d\theta - \beta f)$$

$$= d\alpha^d \zeta \theta \times \left\{ \mathbb{F}_{bin}\left[\left\lfloor \frac{d\theta}{\beta} \right\rfloor; d, 1-\zeta'\right] - \frac{\beta(1-\zeta')}{\theta} \mathbb{F}_{bin}\left[\left\lfloor \frac{d\theta}{\beta} \right\rfloor - 1; d-1, 1-\zeta'\right] \right\}. \quad (10)$$

This is a simpler functional form than equation (9). In equation (9), each counter-party carries a risk of default given by $1 - \alpha^d$, which is endogenously determined by d. Here in equation (10), counterparties are resilient core firms, so counter-party default risk is given by $1 - \zeta'$. As there is no closed-form expression for d^* even in this case, we provide simulations in Figure 12. Firms naturally form more links for larger α and d^* increases. What is striking is that even under weak contagion (small β), even without any cliques (large m and singletons within the periphery), even without any endogenous counter-party risk ($1 - \zeta'$ term in equation (10) instead of the $1 - \alpha^d$ term in equation (9)), the increase in the degree d^* offsets the gain from risk $1 - \alpha$, and systemic risk increases as α increases. Mild network externalities generate the volatility paradox in the core-periphery case as well.

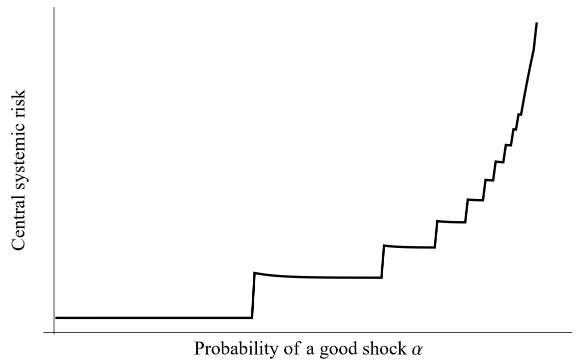

Figure 12: Central volatility paradox persists under weak contagion, $\theta = 1$, $\beta = 2$.

Proof of Proposition A6

Proof. Let $\mathbb{F}$ and $\mathbb{G}$ denote the binomial PDF and CDF. Denote $\tilde{\theta} = \frac{\theta}{\beta}$. From equation (9) we know that

$$V(d) = \beta \sum_{t=0}^{t=d} \mathbb{F}[t, d, 1 - \alpha^d] \left(d\tilde{\theta} - t\right)^+$$

is the payoff of a firm in a clique with $d+1$ firms. If two firms in a clique of order $d^* + 1$ cut a link, their payoff becomes

$$\beta \sum_{t=0}^{t=d-1} \mathbb{F}\left[t, d-1, 1-\alpha^d\right] \left(d\tilde{\theta} - \tilde{\theta} - t\right)^+ < V(d^* - 1) < V(d^*).$$

This cannot be profitable. If two firms in separate cliques of order $d+1$ add their missing link, their payoff becomes

$$W(d) = \beta \sum_{t=0}^{t=d} \mathbb{F}\left[t, d, 1-\alpha^d\right] \left(q(d)\left(d\tilde{\theta} + \tilde{\theta} - t\right)^+ + (1 - q(d)\left(d\tilde{\theta} + \tilde{\theta} - t - 1\right)^+\right)$$

where

$$q(d) = \begin{cases} \alpha^{d+1}\mathbb{G}\left[d\tilde{\theta}, d, 1-\alpha^d\right] & if\ (d+1)\tilde{\theta} < d \\ \alpha^{d+1} & if\ (d+1)\tilde{\theta} \geq d \end{cases}$$

We need to show that $V(d^*) \geq W(d^*)$.

For economy of exposition, in the remainder of the proof, we use $\mathbb{G}_t$ for $\mathbb{G}\left[t, d^*, 1-\alpha^{d^*}\right]$, $\mathbb{F}_t$ for $\mathbb{F}\left[t, d^*, 1-\alpha^{d^*}\right]$, q^* for $q(d^*)$, m for $\lfloor d^*\tilde{\theta} \rfloor$, s for $\lfloor d^*\tilde{\theta} + \tilde{\theta} \rfloor$.

Case 1: $m = s$.

$$\frac{W(d^*)}{\alpha V(d^*)} = \frac{\sum_{t=0}^{t=d^*} \mathbb{F}_t \left(q^* \left(d^*\tilde{\theta} + \tilde{\theta} - t \right)^+ + (1-q^*) \left(d^*\tilde{\theta} + \tilde{\theta} - t - 1 \right)^+ \right)}{\sum_{t=0}^{t=d^*} \mathbb{F}_t \left(d^*\tilde{\theta} - t \right)^+}$$

$$= \frac{\sum_{t=0}^{t=m-1} \mathbb{F}_t \left[\left(d^*\tilde{\theta} + \tilde{\theta} - t \right) - (1-q^*) \right] + \mathbb{F}_m q^* (d^*\tilde{\theta} + \tilde{\theta} - m)}{\sum_{t=0}^{t=m} \mathbb{F}_t \left(d^*\tilde{\theta} - t \right)}$$

$$= \frac{\sum_{t=0}^{t=m} \mathbb{F}_t \left(d^*\tilde{\theta} - t \right) - \mathbb{F}_m (d^*\tilde{\theta} - m) + \mathbb{G}_{m-1}(\tilde{\theta} + q^* - 1) + \mathbb{F}_m q^* (d^*\tilde{\theta} + \tilde{\theta} - m)}{\sum_{t=0}^{t=m} \mathbb{F}_t \left(d^*\tilde{\theta} - t \right)}$$

$$= 1 + \frac{\mathbb{G}_{m-1}(\tilde{\theta} + q^* - 1) + \mathbb{F}_m \left(q^*\tilde{\theta} - (d^*\tilde{\theta} - m)(1-q^*) \right)}{\sum_{t=0}^{t=m} \mathbb{F}_t \left(d^*\tilde{\theta} - t \right)}.$$

If the numerator is negative, we are done. If it is positive:

$$\frac{W(d^*)}{\alpha V(d^*)} \leq 1 + \frac{\mathbb{G}_{m-1}(\tilde{\theta} + q^* - 1) + \mathbb{F}_m \left(q^*\tilde{\theta} - (d^*\tilde{\theta} - m)(1-q^*) \right)}{\mathbb{G}_{m-1}}$$

$$= 1 + \tilde{\theta} + q^* - 1 + \frac{\mathbb{F}_m \left(q^*\tilde{\theta} - (d^*\tilde{\theta} - m)(1-q^*) \right)}{\mathbb{G}_{m-1}}$$

$$\leq \tilde{\theta} + q^* + \frac{\mathbb{F}_m \left(q^*\tilde{\theta} - (d^*\tilde{\theta} - m)(1-q^*) \right)}{\mathbb{F}_{m-1}}$$

$$= \tilde{\theta} + q^* + \frac{d^* + 1 - m}{m} \times \frac{1 - \alpha^{d^*}}{\alpha^{d^*}} \times \left(q^*\tilde{\theta} - (d^*\tilde{\theta} - m)(1-q^*) \right).$$

Define $\epsilon = d^*\tilde{\theta} - m$.

$$\frac{W(d^*)}{\alpha V(d^*)} \leq \tilde{\theta} + q^* + \left(\frac{d^* + 1}{d^*\tilde{\theta} - \epsilon} - 1 \right) \times \frac{1 - \alpha^{d^*}}{\alpha^{d^*}} \times \left(q^*\tilde{\theta} - \epsilon(1-q^*) \right).$$

Consider the function Φ of ϵ keeping all else fixed:

$$\Phi(\epsilon) = \left(\frac{d^* + 1}{d^*\tilde{\theta} - \epsilon} - 1 \right) \left(q^*\tilde{\theta} - \epsilon(1-q^*) \right).$$

$$\Phi'(\epsilon) = (d^* + 1) q^* \tilde{\theta} \frac{-1}{(d^*\tilde{\theta} - \epsilon)^2} - (d^* + 1)(1-q^*) \frac{d^*\tilde{\theta}}{(d^*\tilde{\theta} - \epsilon)^2} + (1-q^*)$$

$$< (d^*+1)q^*\tilde{\theta}\frac{-1}{(d^*\tilde{\theta}-\epsilon)^2} - (1-q^*) + (1-q^*) < 0.$$

So $\Phi(\epsilon)$ is decreasing and maxed at $\epsilon = 0$. That is,

$$\frac{d^*+1-m}{m} \times \left(q^*\tilde{\theta} - (d^*\tilde{\theta}-m)(1-q^*)\right) < \frac{d^*+1}{d^*\tilde{\theta}} \times \left(q^*\tilde{\theta} - \epsilon(1-q^*)\right)$$

and

$$\frac{W(d^*)}{\alpha V(d^*)} \leq \tilde{\theta} + q^* + \left(\frac{d^*+1}{d^*\tilde{\theta}} - 1\right) \times \frac{1-\alpha^{d^*}}{\alpha^{d^*}} \times q^*\tilde{\theta} = \tilde{\theta} + q^* + \left(1 - \tilde{\theta} + \frac{1}{d^*}\right) \times \left(\frac{1-\alpha^{d^*}}{\alpha^{d^*}}\right) \times q^*.$$

Case 1.1: $d^*\tilde{\theta} \geq 1$. In this case, $1 - \tilde{\theta} + \frac{1}{d^*} \leq 1$ so that

$$\frac{W(d^*)}{\alpha V(d^*)} \leq \tilde{\theta} + q^* + \left(\frac{1}{\alpha^{d^*}} - 1\right) \times q^* = \tilde{\theta} + \frac{q^*}{\alpha^{d^*}} \leq \tilde{\theta} + \alpha \leq \frac{1}{\alpha}.$$

Case 1.2: $d^*\tilde{\theta} < 1$. Then $m = s = 0$. Since $s = 0$, $(d^*+1)\tilde{\theta} < 1$. Optimal d^* can't be zero, so $(d^*+1)\tilde{\theta} < 1 \leq d^*$, hence $q(d^*) = \alpha^{d^*+1}\mathbb{G}_m = \alpha^{d^*+1}\mathbb{G}_0 = \alpha^{d^*+1}\mathbb{F}_0 = \alpha^{d^*+1+(d^*)^2}$. Then $W(d^*) = \beta\alpha^{d^*+1}\mathbb{F}_0 q^*(d^*\tilde{\theta} + \tilde{\theta}) = \beta\alpha^{2d^*+2+2(d^*)^2}(d^*\tilde{\theta} + \tilde{\theta})$. On the other hand $V(d^*+1) = \beta\alpha^{d^*+1+(d^*+1)^2}(d^*\tilde{\theta} + \tilde{\theta})$. Since $d^* \geq 1$, we have $2d^* + 2 + 2\left(d^*\right)^2 \geq d^* + 1 + (d^*+1)^2$ so that $V(d^*+1) \geq W(d^*)$, which implies $V(d^*) \geq W(d^*)$.

Case 2: $m = s - 1$.

Case 2.1: $(d^*+1)\tilde{\theta} < d^*$.

$$\frac{V(d^*)}{\beta\alpha^{d^*}} - \frac{W(d^*)}{\beta\alpha^{d^*+1}} = \sum_{t=0}^{t=d^*} \mathbb{F}_t\left(d^*\tilde{\theta} - t\right)^+ - \left[\sum_{t=0}^{t=d^*} \mathbb{F}_t\left(q^*\left(d^*\tilde{\theta} + \tilde{\theta} - t\right)^+ + (1-q^*)\left(d^*\tilde{\theta} + \tilde{\theta} - t - 1\right)^+\right)\right]$$

$$= \sum_{t=0}^{t=m} \mathbb{F}_t\left(d^*\tilde{\theta} - t\right) - \left[\sum_{t=0}^{t=s-1=m} \mathbb{F}_t\left(d^*\tilde{\theta} + \tilde{\theta} - t - 1 + q^*\right) + \mathbb{F}_s q^*(d^*\tilde{\theta} + \tilde{\theta} - s)\right]$$

$$= \mathbb{G}_m(1 - \tilde{\theta} - q^*) - \mathbb{F}_{m+1}q^*(d^*\tilde{\theta} + \tilde{\theta} - s).$$

$d^*\tilde{\theta} < m + 1 = s$ so $d^*\tilde{\theta} + \tilde{\theta} - s < \tilde{\theta}$. Also $(d^*+1)\tilde{\theta} < d^*$, so that $q(d^*) = \alpha^{d^*+1}\mathbb{G}_m$. Also $\mathbb{F}_{m+1} < 1 - \mathbb{G}_m$. Plug all these in:

$$\frac{V(d^*)}{\beta\alpha^{d^*}} - \frac{W(d^*)}{\beta\alpha^{d^*+1}} > \mathbb{G}_m\left(1 - \tilde{\theta} - \alpha^{d^*+1}\mathbb{G}_m - (1-\mathbb{G}_m)\alpha^{d^*+1}\tilde{\theta}\right).$$

$$> \mathbb{G}_m \left(1 - \tilde{\theta} - \alpha^{d^*+1}\mathbb{G}_m(1-\tilde{\theta}) - \alpha^{d^*+1}\tilde{\theta}\right) > \mathbb{G}_m \left(1 - \tilde{\theta} - \alpha^{d^*+1}(1-\tilde{\theta}) - \alpha^{d^*+1}\tilde{\theta}\right)$$

$$= \mathbb{G}_m \left(1 - \tilde{\theta} - \alpha^{d^*+1}\right) \geq 0.$$

That is $V(d^*) > \frac{1}{\alpha}W(d^*) > W(d^*)$.

Case 2.2: $(d^*+1)\tilde{\theta} \geq d^*$. Then $s = d^*$, so that $m = d^* - 1$. Note that $(d^*+1)\tilde{\theta} \geq \tilde{\theta}$ implies $(d^*+1)\alpha^2 \leq 1$.

Case 2.2.1: If $d^* \geq 3$, then

$$W(d^*) < \beta\alpha^{d^*+1}(d^*\tilde{\theta} + \tilde{\theta}) \leq \beta\alpha^4(d^*+1)\tilde{\theta} \leq \beta\alpha^2\tilde{\theta} = V(1) \leq V(d^*).$$

Case 2.2.2: If $d^* = 2$, then $s = 2$ so $\tilde{\theta} > \frac{2}{3} > \frac{1}{2}$. Then

$$\frac{V(2)}{\beta\alpha^2} = \alpha^4 2\tilde{\theta} + 2\alpha^2(1-\alpha^2)(2\tilde{\theta} - 1)$$

$$= 2\tilde{\theta}\left[2\alpha^2 - \alpha^4\right] - 2\alpha^2\left(1 - \alpha^2\right) < 2\tilde{\theta}\left[2\alpha^2 - \alpha^4\right] - 2\alpha^2\tilde{\theta}$$

$$= 2\tilde{\theta}\left[\alpha^2 - \alpha^4\right] < \frac{\tilde{\theta}}{2} < \tilde{\theta} = \frac{V(1)}{\beta\alpha^2}$$

which is a contradiction. $d^* = 2$ is not possible in this case.

Case 2.2.3: If $d^* = 1$, then $s = 1$ and $\tilde{\theta} > \frac{1}{2}$.

$$\frac{W(1)}{\beta\alpha^2} = \alpha\left(\alpha^2 2\tilde{\theta} + (1-\alpha^2)(2\tilde{\theta} - 1)\right) + (1-\alpha)\left(\alpha^2(2\tilde{\theta} - 1)\right)$$

$$= 2\tilde{\theta}\left[\alpha + \alpha^2 - \alpha^3\right] - \left[\alpha + \alpha^2 - 2\alpha^3\right].$$

$$\frac{V(1) - W(1)}{\beta\alpha^2} = 2\tilde{\theta}\left[\frac{1}{2} - \alpha - \alpha^2 + \alpha^3\right] + \left[\alpha + \alpha^2 - 2\alpha^3\right].$$

If the term in the first bracket is positive, we are done. If it is negative, then replace insert $\tilde{\theta} < 1 - \alpha^2$.

$$\frac{V(1) - W(1)}{\beta\alpha^2} > 2\left(1 - \alpha^2\right)\left[\frac{1}{2} - \alpha - \alpha^2 + \alpha^3\right] + \left[\alpha + \alpha^2 - 2\alpha^3\right]$$

$$= (1-\alpha)\left(1 - \alpha^2\right) \geq 0.$$

□

B.3 Node shocks and systemic risk

As mentioned in Section 3.3, changing the probability of good node shocks, ζ and ζ', impacts the network as well. The comparative statics in ζ and ζ' are less interesting and the effects are summarized in Table 2. The only interesting case is when ζ increases for large core (large α). Then, systemic risk has an inverse U-shape displayed in Figure 13. The reason that systemic risk initially increases is similar to the the peripheral volatility paradox introduced in Section 2.3. The reason systemic risk starts to decline after a threshold of ζ is that the risk of link shocks $1-\alpha$ imposes an upper bound on the number links that peripheral firms want to form no matter how large ζ can be. Accordingly, when the periphery reaches this "satiation" point, the cliques do not expand any further and systemic risk stars falling as ζ increases.

Systemic	w.r.t.:	ζ (periphery node-shock)			ζ' (core node-shock)		
risk	source:	Central	Peripheral	Combined	Central	Peripheral	Combined
$d^* \leq m$ / small α		Constant	Decr.	Decr.	Decr.	Incr.	Decr.
$d^* > m$ / large α		Constant	Inverse-U	Inverse-U	Decr.	Constant	Decr.

Table 2: Central and peripheral systemic risk in node shock risk

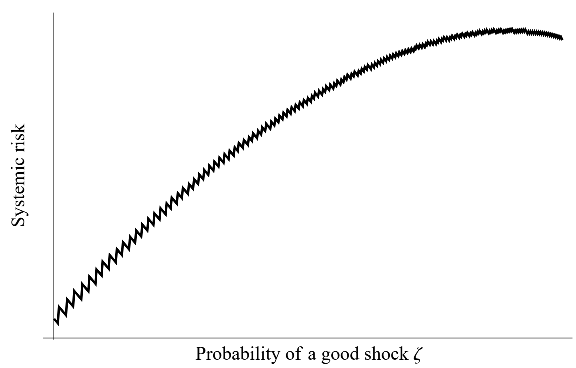

Figure 13: Volatility paradox with respect to the probability of a good node shock for the periphery