Network Hazard

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Abstract

This paper introduces a novel form of moral hazard specific to networks and illustrates this concept using simple models from coordination games, epidemics, supply chains, and financial networks. In these models, agents form beneficial links that also propagate costly contagion. Endogenously, second-order contagion risk constrains the concentration of connections around central agents. Protective measures against contagion, such as vaccines, subsidies, or bailouts, mitigate contagion risk, subsequently increasing concentration. However, if these protective measures are imperfect or costly, shocks to central agents can result in greater harm and increased welfare variance, as evidenced in disease outbreaks, aggregate volatility, or financial crises.

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1 Introduction

In recent years, networks have gained significant prominence across various markets. The emergence of the internet and social media has facilitated the expansion of social networks, enabling the rapid dissemination of information and disrupting traditional marketing practices. Additionally, advancements in transportation and communication technologies have contributed to the globalization of supply chains, while increased mobility and income growth have broadened opportunities for global tourism and labor mobility. Furthermore, contemporary central banking post the gold standard, the relaxation of branching restrictions, and the growth of computational power have resulted in increasingly complex yet accessible financial contracts and systems.

The advantages of a more interconnected world are accompanied by notable drawbacks. The interdependence of agents within a network creates vulnerabilities that may be transmitted through links within the network, raising concerns about contagion. For example, natural disasters can lead to downstream supply chain disruptions, infectious diseases can proliferate through personal connections, and banks can experience cascading failures.

The granular nature of networked interactions differentiates these negative externalities from those found in standard economic models. A bottleneck link (as in Manea (2021)) or a central agent (as in Galeotti and Goyal (2010)) can disproportionately influence economic outcomes, provided that they are not immune to contagion. Consequently, comprehending the formation of networks and the interplay between economic fundamentals and the topological properties of endogenous networks is crucial for predicting economic outcomes. For instance, endogenous superspreader events posed a significant threat to public health during the COVID-19 pandemic. Certain public figures exert extensive influence over polarized societies, as evidenced by social media. Large, interconnected banks were at the heart of the 2008 financial crisis, which caused unprecedented turmoil in nearly a century.

In light of the intricate and far-reaching implications of contagion, authorities have sought to mitigate their effects. One policy tool involves intervening in link formation, such as imposing quarantines to prevent socialization or imposing capital adequacy to limit banks' derivative exposures to one another. Although theoretically effective, such regulation often proves insufficient, prompting authorities to also intervene in contagion processes. For example, vaccines can reduce transmission probabilities across links, while bailouts can provide capital to institutions already exposed to default risks.

This study investigates the consequences of interventions in contagion when links are endogenously formed prior to contagion. The core concept is that the availability of protective measures may cause agents to become less cautious, resulting in networks that are more susceptible to contagion. If protective measures are imperfect or impose significant social costs, the outcome is a form of collective moral hazard, referred to as network hazard. The presence of protective measures may lead to a higher number of infections ex-ante or increased social costs ex-ante due to an endogenously more vulnerable network. For instance, more effective or widely available vaccines may encourage larger social gatherings, potentially leading to superspreader events with a smaller probability, but resulting in a greater number of infections. In the context of bailouts, financial networks may become more interconnected and diversified around core banks, which may default with a small probability but have a larger overall impact on peripheral banks.

Network hazard can manifest in numerous forms through the various types of effects it can have on the network in question. To demonstrate a particularly pertinent and novel type of network hazard, we employ the most elementary example. Consider a two-sided market, such as opposite genders in a matching market confronted with the risk of an infectious disease (or upstream and downstream producers in a supply chain, or insurance sellers and buyers in a financial network). Agents can get infected exogenously, but they can also transmit infection through links formed. Let one side be represented by a single agent A, and the other side be comprised of agents B and C, as depicted in Figure 1a. Assuming B and C are identical, they each determine whether to establish a connection with A for a private gain. A accepts all connections.

Depending on the efficacy of a protective measure the network can be one of three in Figure 1*a*-*c* without loss of generality.

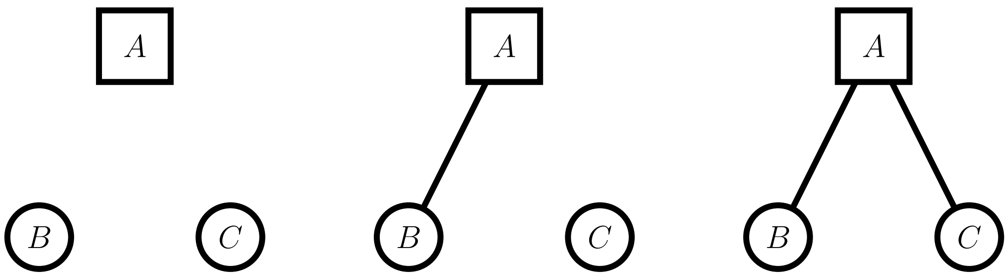

Figure 1*a*
Low efficacy
protective measure.

Figure 1*b*
Medium efficacy
protective measure.

Figure 1*c*
High efficacy
protective measure.

Figure 1: Stable network

As the protective measure gets more effective against the disease, the endogenous network changes from Figure 1*a* to 1*b*, then to 1*c*. To illuminate the key idea, think of a status-quo where the protective measure efficacy is just below the cutoff at which network 1*c* would be formed. Instead, 1*b* is formed. This means B is content to have a connection with A, signifying that B is willing to accept the risk of contagion originating from A and infecting B, for the private payoff of the connection with A. Although B and C are symmetric, C does not form a connection with A, specifically due to B's existing connection to A. The risk of contagion originating (exogenously) at B, subsequently infecting A, and then infecting C is not a risk C is willing to assume on top of the risk of contagion originating from A and infecting C. In other words, the risk of second-order contagion from B *through* A disciplines C into not connecting to A. Now increase the efficacy of the protective measure slightly, up to the cutoff point where the network changes from Figure 1*b* to Figure 1*c*. This is exactly the point at which C is not concerned enough with contagion through A due to the protective measure. Infection probabilities decrease slightly on the intensive margins, but all agents are now exposed to more agents on the extensive margin. Hence the infection probability of each agent increases. Importantly, B and C are now both exposed to contagion that originates at A, making the infection of B and C more correlated. This increases variance of the number of infections, not only its mean.[1] Regarding welfare, C's payoff increases only slightly as he is indifferent between Figure 1b and 1c at the cutoff, but B's payoff decreases considerably due to the second-order exposure to C through A in Figure 1c. Thus welfare decreases.[2]

The core argument presented in this simple example can be applied more broadly.[3] To demonstrate, this paper introduces four elementary models of distinct settings to elucidate the concept of network hazard.

Related literature and the structure of the paper Section 2 explores coordination games building upon a simplified variant of Galeotti, Golub and Goyal (2020). Morris (2000) describes the correspondence between contagion and coordination games, while Galeotti, Golub and Goyal (2020) investigate interventions on fixed networks where a coordination game is played. We add network formation to further investigate the interplay between network formation, coordination games played on networks, and interventions.

Section 3 examines epidemics in a standard independent cascades model extended to incorporate network formation. A companion paper, Celdir and Erol (2023), utilizes a less sophisticated framework than presented here to guide their empirical analysis, and document that foot traffic to central locations and infection rates increase as vaccination rates rise.

Section 4 introduces a model of supply chains where downstream firms choose their upstream suppliers. Elliott, Golub and Leduc (2022) and Acemoglu and Tahbaz-Salehi (2020) study how minor shocks can be amplified throughout the network, resulting in fragility and volatility. This paper contributes the notion that networks more susceptible to fragility may emerge in response to

[1]This can have further consequences when hospital capacity is taken into account, as was the case in COVID-19 pandemic..

[2]We neglect A's payoff due to lack of discipline in this simple example. A's payoff parameters can be scaled up and down without affecting the network.

[3]In independent cascades such as infectious diseases, more links increase contagion risk. In threshold contagion models, law of large numbers can wash out idiosyncratic risk if a large number of links are preferred by agents. Nevertheless, network hazard manifests under threshold contagion when shocks are correlated.

the anticipation of interventions. Liu (2019) studies macroeconomic distortions in endogenous input-output relationships between sectors in face of selective industrial policies. This paper considers resilience consequences of interventions.

Section 5 introduces a model of derivative contracts. Elliott, Golub and Jackson (2014) and Acemoglu, Ozdaglar and Tahbaz-Salehi (2015) provide canonical models of financial contagion. Elliott, Georg and Hazell (2021) and Acemoglu, Ozdaglar and Tahbaz-Salehi (2015) examine contagion and the formation of financial networks.[4] A companion paper, Erol (2019), analyzes some manifestations of network hazard in a stylized setup. The model here incorporates more detailed financial contracts and costly bailouts.

Testing the broader theory is challenging, as pandemics, tsunamis, or financial crises are uncommon (yet devastating) events, and the anticipation of interventions is often unobserved. Celdir and Erol (2023) document that, as vaccines began rolling out in December 2020, foot traffic to central locations such as gyms, restaurants, and airports started to increase gradually, culminating in a massive outbreak in January 2022. In the case of financial networks, counterfactuals regarding the anticipation of bailouts are nearly nonexistent. Anderson, Erol and Ordonez (2020) demonstrate that the concentration of short-term borrowing increased after the establishment of the Federal Reserve System, which also served as the lender of last resort. These findings align with the testable predictions of the theory.

2 Coordination games and interventions

Model. Here we study network formation and monetary interventions using a simple variant of the model in Galeotti et al. (2020) OA3.4. Agents first form undirected links which make up the network denoted e. A link between agents i and j is indicated by $e_{ij} = e_{ji} \in \{0, 1\}$. If $e_{ij} = 1$ we say i and j are connected and we denote $e_{ii} = 0$. The cost of $e_{ij} = 1$ is c_{ij} for i. After the

[4]Erol and Vohra (2022) also studies network formation and contagion. Erol and Ordonez (2017) studies interventions with network formation rather than interventions with contagion.

formation of the network, shocks $\tau_i > 0$ realize. Each agent i simultaneously chooses $a_i \in \{0, 1\}$ with complete information. Given an action profile a, the payoff of i in the coordination game is given by $v_i(a) = a_i \sum_j (\beta_{ij} a_j - \tau_i) e_{ij}$. The ex-post payoff of i is

$$u_i(a, g) = \sum_j (a_i (\beta_{ij} a_j - \tau_i) - c_{ij}) e_{ij}$$

It is assumed that agents play the best[5] Nash equilibrium of the coordination game and they form (pairwise) stable networks given the expected continuation payoffs.

We focus on a simple case. There are two possible shocks $\tau_i \in \{g_i, b_i\}$. The good shock $g_i > 0$ has probability α_i and the arbitrarily large bad shock $b_i > 0$ has probability $1 - \alpha_i$. In terms of the risk an agent imposes on its counterparties, there is one risky agent denoted r, and two ex-ante identical safe agents s_1 and s_2. A link between a safe agent and the risky agent gives benefit β_s to the safe agent and β_r to the risky agent where $\beta_r > \beta_s$. Also, $\alpha_s > \alpha_r$. We take $\beta_s > g_s$ and $\beta_r > g_r$ so that links with the opposite can be formed rationally. The benefit of link between two safe agents to be zero but its cost is positive, so safe agents do not form a link with each other. Links cost 0 to the risky agent and $c > 0$ to safe agents so network formation is driven mainly by the incentives of safe agents. Assume $2\alpha_s > 1$, denote $\kappa \equiv \frac{c}{\beta_s - g_s}$, and $\omega \equiv 2g_r - \beta_r$.

Proposition 1. *The unique stable network is given as follows. Safe agents do not have links with each other.*

Under $\omega > 0$, both s agents have links with r if $\alpha_r \alpha_s^2 > \kappa$, only one s agent has a link with r if $\alpha_r \alpha_s > \kappa > \alpha_r \alpha_s^2$, and there are no links if $\kappa > \alpha_r \alpha_s$.

Under $\omega < 0$, both s agents have links with r if $\alpha_r \alpha_s > \kappa$, and there are no links if $\kappa > \alpha_r \alpha_s$.

The most relevant case for network hazard is $\omega > 0$ and $\alpha_r \alpha_s > \kappa > \alpha_r \alpha_s^2$. Under $\omega > 0$, the benefit from one link is insufficient to play $a_r = 1$. This

[5]Top element of the lattice of Nash equilibria.

means that if r has two connections and s_k gets a bad shock, r plays 0, which then pushes $s_{k'}$ to play 0 as well. This means safe agents are exposed to second-order counterparty risk of they both connect with r.

Welfare and interventions. Imagine a principal who observes the network and the shocks, and can commit a transfer scheme conditional on actions to be taken to maximize welfare. Given the transfer scheme $t_i(a_i|e,\tau)$, agent i's payoff in the coordination game is $v_i(a|t,e,\tau) = t_i(a_i|e,\tau) + \sum_j a_i (\beta_{ij} a_j - \tau_i) e_{ij}$. The welfare cost of transfers is $\kappa \geq 1$. The principal's objective is to maximize welfare. We take $\kappa \downarrow 1$ to rule our redundant transfers. This is equivalent selecting the minimal transfer scheme among the optimal ones. Ex-post welfare is then given by[6]

$$w(a|t,e,\tau) = -\sum_i t_i(a_i|e,\tau) + \sum_i v_i(a|e,\tau) = \sum_i \sum_j a_i (\beta_{ij} a_j - \tau_i) e_{ij}$$

Denote $\omega' = \beta_s - g_s$.

Proposition 2. *Under $\omega + \omega' < 0$ or $\omega' > 0$ there are no transfers. Accordingly, the unique stable network is the same with the one in the absence of interventions.*

Under $\omega' > -\omega > 0$, $t_r(1|e,\tau) = -\omega$ if r has two links, for one $k \in \{1,2\}$ r and s_k have good shocks and $s_{k'}$ has a bad shock. All other transfers are 0 in all other cases of shock realizations and networks. The unique stable network involves two links if $\alpha_r \alpha_s > \kappa$ and no links if $\alpha_r \alpha_s < \kappa$.

The precise role of an optimal and non-zero transfer is to stop second-order counterparty risk. There are no interventions if there is no contagion. Given that second-order counterparty risk is eliminated by interventions, the network formed involves two links. But interventions do not alter first-order counterparty risk.

[6]Given that transfers are conditional individual actions, the best Nash equilibrium is still well-defined.

Proposition 3. *Assume $\omega' > -\omega > 0$ and $\alpha_r\alpha_s > \kappa > \alpha_r\alpha_s^2$. Variance of welfare is larger in the presence of interventions than in the absence of interventions. The change in the expectation of welfare can be positive or negative depending on parameters.*

Network hazard arises in the case of $\omega' > -\omega > 0$ and $\alpha_r\alpha_s > \kappa > \alpha_r\alpha_s^2$. In the absence of interventions it is individually rational for each safe agent to have a link with the risky agent, but only if the other safe agent does not have a link with the risky agent. Interventions remove this discipline and both safe agents form links with risky agent. Then both safe agents are subject to first-order counterparty risk, which is now correlated due to the anticipation of interventions. This induces high variance in welfare through the increased first-order counterparty risk.

3 Epidemics and protective measures

Model. Think of a population of agents that obtain benefits from socialization but each connection results in the risk of disease transmission. For agents i and j, v_{ij} is benefit i has from having a connection with j. N_i denotes the set of agents that i has connections with and $d_i = |N_i|$ is i's degree. The interaction payoff of i us $-d_i^2 + \sum_{j\in N_i} v_{ij}$.

There is an infectious disease. Each agent has η probability of being infected externally. Each connection entails risk of transmission. The base transmission probability is τ_0 if one agent in the connection is infected and the other is not. There is a protective measure that reduces the transmission probability by a factor $m < 1$, down to $\tau = m\tau_0$.[7] Getting infected has cost κ.

The model can speak to various other environments. Let each agent have a type $t \in \{a, b\}$. Connections can reflect friendships and types can capture tastes for socialization when $v_{ij} > 0$ for all agents. The environment can also

[7]Costs of using the protective measure is assumed away so all agents adopt the protective measure.

describe two-sided matching by assuming that within group values are zero: $v_{ij} = 0$ if $i, j \in t \in \{a, b\}$. Connections can be education while types are teachers and students. Connections can be shopping while types are customers and employees of grocery stores. Connections can be sexual partnerships or encounters and types can reflect sexual orientation.

We focus on a simple case to highlight network hazard but high level insights hold more generally. Consider two a-type agents, a_1, a_2, and two b-type agents, b_1, b_2. The value of within-group connections are 0. The value of a cross-group connection for an a-type agent is sufficiently large so that a-types welcome all connections to b-types. Let $v_1 \equiv v_{b_k a_1} > v_2 \equiv v_{b_k a_2}$ for both $k = 1, 2$. This means b-types have the same preferences over a-type agents they prefer to connect with a_1 over a_2. We call a_1 the attractive agent and a_2 the unattractive agent. Assume $3 \geq v_1 > v_2$ so that b-type agents want at most one connection.

Stability. We use the notion of (strong) stability. Absent the infectious disease, the unique stable network involves both b-types having connections with a_1. The potential spread of the disease can alter this. When both b-types are connected with a_1, b_k can get infect a_1, who can then infect $b_{k'}$. This is second-order counterparty risk. The efficacy of the protective measure m plays a key role in determining the stable network.

Proposition 4. *There are thresholds $m_3 \leq m_2 < m_1$ such that the unique stable network is given as follows. If the protective measure is effective, $m < m_3$, both b-type agents are connected with the attractive agent a_1. If the protective measure is mildly effective, $m_3 < m < m_2$, one b-type agent is connected with the attractive agent a_1 and the other is connected with the unattractive agent a_2. If $m_2 < m < m_1$, one b-type agent is connected with the attractive agent a_1 and the other b-type agent is not connected to any agent. If $m_1 < m$, there are no connections.*[8]

[8] $m_1 \equiv m_1^*$, $m_2 \equiv \max\{m_2^*, m^*\} \leq m_3 \equiv \min\{m^{**}, m^*\}$ where $m_i^* \equiv \frac{v_i - 1}{\kappa(1-\eta)\eta\tau_0}$, $m^* \equiv \frac{\sqrt{1+4\frac{v_1-1}{\kappa\eta}}-1}{2(1-\eta)\tau_0}$, $m^{**} \equiv \frac{\sqrt{v_1 - v_2}}{\sqrt{\kappa\eta}(1-\eta)\tau_0}$. Note that we either have $m_2^* < m^* < m^{**}$ or $m_2^* > m^* > m^{**}$ or $m_2^* = m^* = m^{**}$. So $m_3 = m_2 \iff m_2^* \leq m^{**} \iff v_2 - 1 \leq \sqrt{\kappa\eta(v_1 - v_2)}$

As m decreases meaning that the protective measure gets more effective, the network of connections becomes more interconnected. For b-type agents, at the margin m_1, it becomes individually rational to connect with the attractive agent. At m_2, it becomes individually rational to connect with the unattractive agent as well. However, second-order counterparty risk is still high enough that it is not desirable to connect with the attractive agent at the same time. At m_3, second-order counterparty is low enough that both b-type agents connect with the attractive agent. These are shown in Figure 2

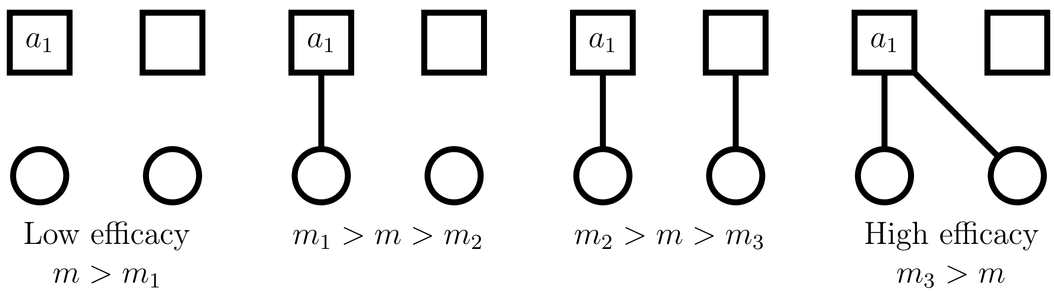

Figure 2: Stable network

Proposition 5. *Let $\eta < 1/4$. As the protective measure gets more effective (m goes down) the expectation and variance of the number of infections increase at points $m \in \{m_1, m_2, m_3\}$ at which the stable network changes.*

We report the distribution of the number of infections, its mean, and its variance in the proof of Proposition 5. The increases at m_1 and m_2 can be expected. The number of connections increase at these margins. The switch at m_3 does not introduce a new connection. A b-type agent, say b_i, switches its connection with a_2 to a connection with a_1. This is because the protective measure becomes effective enough that b types are less concerned with being infected "by each other" via a_1, the attractive a-type agent.[9] This effect increases the expected number of infections. Additionally, now that both b-type agents are exposed to the exogenous infection risk of the attractive agent a_1, variance of infections also increase. These are illustrated in Figures 3 and 4.[10] The vertical lines correspond to m_3.

[9] This happens in the case of $v_2 - 1 \leq \sqrt{\kappa\eta(v_1 - v_2)}$.

[10] $\eta = 0.1, \tau_0 = 0.75, v_1 = 3, v_2 = 2.5, \kappa = 40$

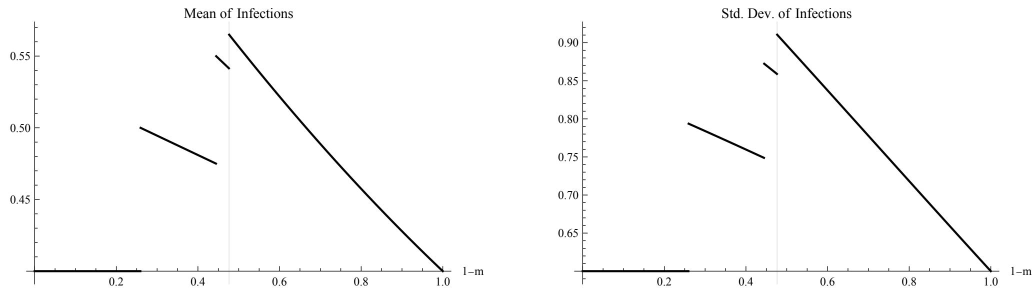

Figure 3: Mean and standard deviation of the number of infections

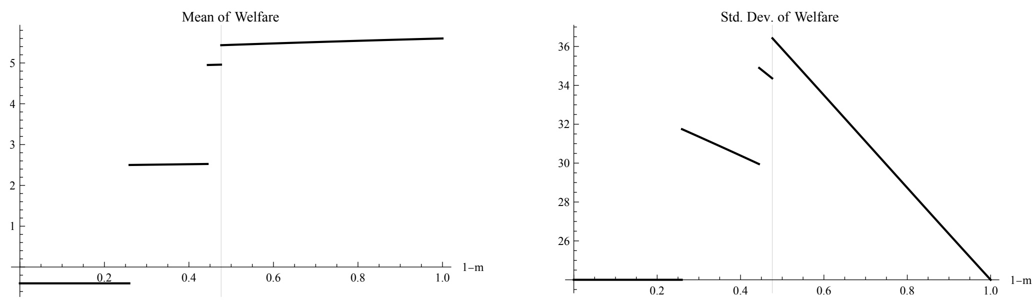

Figure 4: Mean and standard deviation of welfare

There are numerous avenues for future research. Limited hospital capacity posed a significant challenge in combating COVID-19. Network hazard contributes to greater variance in infection rates, underscoring the importance of considering network hazard in the context of constrained hospital resources. Another essential aspect to explore is the integration of multiple protective measures, such as masks that reduce transmission and vaccines that alleviate disease severity.

4 Supply chains and subsidies

Model. There are two upstream firms $U = \{u_1, u_2\}$. They produce substitute but differentiated products. There are two downstream firms $D = \{d_1, d_2\}$. Each downstream firm picks a specific production technology that is compatible with the input from only one of the suppliers. Switching costs are high enough that once an upstream supplier's technology is chosen by a downstream firm, the supplier becomes the monopoly supplier of the downstream firm.

Upstream $u \in U$ uses specific external input which has price k_u given by k with probability α_u and k' with probability $1 - \alpha_u$ where $k' > k$. We assume $\alpha_{u_1} > \alpha_{u_2}$. Downstream $d \in D$ uses a specific external input and an internal input which are perfect complements. The specific external input is has price c_u given by c with probability δ and c' with probability $1 - \delta$ where $c' > c$. The internal input is the output of d's chosen supplier/upstream firm. Each $d \in D$ is a monopolistic seller of its own output to a consumer who has value p, all of which is extracted by d. The value p is shared by d and its supplier. The supplier gets p_U share whereas d gets $p_D = p - p_U$. Equivalently, the price of internal input is p_U, which is fixed for simplicity. We assume $\delta p_U > k$ and $p_D > c$ so production is efficient for firms when costs are low.

First, downstream firms choose their technologies/suppliers simultaneously. Then costs of upstream firms are realized independently. Then upstream firms build their inventory. Then costs of downstream firms are realized independently. Then downstream firms purchase inputs and produce. Then consumers buy outputs of the downstream firms.

When an upstream firm has two buyers, total demand for an internal input can exceed its supply. In particular, if u has two downstream buyers but produces 1 instead of 2, we assume that the supplier sells on a fist-come-first-serve basis. Each downstream buyer has probability 1/2 of being supplied.

We assume $k', c' > p$. Then firms high costs, c' or k', do not produce. Denote $D_u \subset D$ the downstream buyers of $u \in U$ and $u_d \in U$ the supplier of $d \in D$. Then the ex-post payoff of u is $v_u = 0$ if it has high cost and

$$v_u = -k_u q_u + p_U \sum_{d \in D_u} q_d$$

if it has low cost. Regarding d, if either u_d or d has high cost, or u_d does not supply d, d's ex-post payoff is $v_d = 0$. Otherwise it is

$$v_d = (-c + p_D)\, q_d$$

Equilibrium. At the stage of choosing their suppliers, downstream firms face first-order counterparty risk in that their supplier may have high cost and not produce. Upstream firms also face first-order counterparty risk as their downstream buyers may have high costs and not demand any input. Since upstream firms must first build inventory, the first-order counterparty risk can induce an upstream firm with two buyers to produce only 1 unit instead of 2 even when it has two buyers. This creates second-order risk for downstream firms by reducing their probability of being supplied.

Proposition 6. *If* $\frac{\alpha_{u_2}}{\alpha_{u_1}} > 1 - \delta + \frac{\delta}{2}$ *and* $k > \delta^2 p_U$*, downstream firms choose separate suppliers. Off-the-path, if both downstream firms choose* u *and* u *has low cost, it produces* 1 *and supplies at most one downstream firm.*

If $\frac{\alpha_{u_2}}{\alpha_{u_1}} < 1 - \delta + \frac{\delta}{2}$ *and* $k > \delta^2 p_U$*, both downstream firms choose* u_1*. If* u_1 *has low cost, it produces* 1 *and supplies at most one downstream firm.*

If $k < \delta^2 p_U$*, both downstream firms choose* u_1*. If* u_1 *has low cost, it produces 2 and supplies each downstream firm that has a low cost.*

The first case is the most relevant. As u_1 entails smaller first-order counterparty risk, it is a preferred supplier. However by $k > \delta^2 p_U$, the cost of building sufficient inventory to supply both downstream firms is high relative to the expectation of demand. Even if u_1 were chosen by both downstream firms, u_1 would produce only 1. Then each downstream firms faces the risk of not being supplied if the other downstream firm has low cost. This second-order counterparty risk is costly for downstream firms. By $\frac{\alpha_{u_2}}{\alpha_{u_1}} > 1 - \delta + \frac{\delta}{2}$, the first-order counterparty risk entailed by u_2 is low so that one downstream firm prefers high first-order counterparty risk over high second-order counterparty risk and chooses u_2. Depending on the economic costs of inputs, second-order counterparty risk creates an inefficiency by increasing the first-order counterparty risk.

Welfare and interventions. In identifying welfare we specify the economic costs of inputs. Internal inputs are those that downstream firms buy from upstream firms. These are transfers and do not count towards the ex-post welfare criterion. The external costs of firms can be varied in a similar vein. Some costs of firms are payments to agents within the economy who produce these inputs at low cost. For example, wages for high skill labor, debt to banks, patent rentals, solar energy are produced using another fundamental input that has relatively low marginal cost to produce. This situation can arise in imperfectly competitive segments of the economy. The other costs of the firm are payments for imported goods or for inputs that are produced domestically at a relatively high cost. Domestically produced inputs that are sold at competitive markets would entail economic costs close to the cost for the firm. Let $e_{u/d}$ be the economic cost of the external input of u/d. Then denoting $q_{u/d}$ the quantity produced, ex-post welfare is w

$$w = \sum_{d \in D} (p - e_d)\, q_d - \sum_{u \in U} e_u q_u$$

The source of the shock to a firm's costs can have "real" economic roots or can be "financial" in nature. For example inflation in specific sectors can increase the cost of corresponding inputs for firms without necessarily making production inefficient. Alternatively mergers and acquisitions can alter competition in a sector and alter prices without necessarily increasing the production costs of inputs. To highlight network hazard, we take a high cost for downstream firms to be a financial shock in nature and to upstream firms to be real: $e_d = 0$, $c_u \in \{c, c'\}$ whereas $e_u = k_u \in \{k, k'\}$.

A bad shock to a downstream firm increasing its costs can result in inefficiently low production. The government can provide subsidies to corresponding firms to promote efficient production. We consider a government that, following each shock, can offer a subsidy to the firm hit by the shock for its external inputs. Denote $s_{u/d}$ the subsidy to u/d per unit of output to subsidize its cost of external inputs. The total cost of transfers is $\sum_{u \in U} s_u q_u + \sum_{d \in D} s_d q_d$ assuming away distortionary effects of transfers for simplicity. To discipline redundant transfers focus on minimal subsidies that implement the efficient outcome. This specification also provides robustness to small distortionary costs of transfers.

Proposition 7. *In the presence of subsidies, each downstream firm $d \in D$ receives $s_d = c' - p_D$ if its supplier has low cost. Upstream firms do not receive subsidies. Both downstream firms choose u_1.*

Since $k' > p$, u and its buyers do not receive a subsidies if $k_u = k'$. Downstream firms receive subsidies whenever they have high costs that would hinder their production provided that their supplier has low cost. Given that the upstream firms do not face first-order counterparty risk, they produce enough to supply the equilibrium demand of their downstream buyers. This eliminates second-order counterparty risk. Since first-order counterparty risk that downstream firms face is not altered but second-order counterparty risk is eliminated, both downstream firms choose the same supplier, u_1. Next we describe the welfare consequences of such interventions and network reactions. For clarity we take α_{u_1} and α_{u_2} to be close.

Proposition 8. *Suppose that $\alpha_{u_2} \approx \alpha_{u_1} = \alpha < 1 - \frac{\delta}{2(1-\delta)}$ and $k > \delta^2 p_U$. Expectation and variance of welfare is larger in the presence of subsidies than in the absence of subsidies.*

Expected welfare naturally increases with interventions. What prevented downstream firms from choosing the supplier with lower first-order counterparty risk was second-order counterparty risk in the first place. Subsidies remove this inefficiency. On the other hand, both downstream firms choose the same supplier with turns the idiosyncratic shock of the supplier into a source of aggregate volatility.

5 Derivatives and bailouts

Model. There is one investment bank n and two commercial banks c_1, c_2. Commercial banks have $1 + p$ deposits borrowed from their corresponding depositors and each has a project that needs 1 unit of investment. Successful projects have return rate r_c with probability σ_c and 0 with probability $1 - \sigma_c$. Each commercial bank has illiquid assets worth $l_c\lambda_c$ at maturity. Illiquid assets are divisible. Early liquidation of illiquid assets that is worth $\lambda_c > 1$ at maturity recovers 1. We assume $l_c > 1 + p$ so deposits are secured and depositors do not ask for interest.

The investment bank sells insurance and invests the proceeds into a financial instrument which has return rate r_n with probability σ_n and 0 otherwise with probability $1 - \sigma_n$. The investment bank has illiquid assets worth $l_n\lambda_n$ at maturity. Similar to commercial banks, early liquidation of illiquid assets worth $\lambda_n > 1$ at maturity recovers 1. We simplify the analysis by fixing prices. Insurance contracts cost p, and entitles the corresponding commercial bank to a claim p' if its project fails. If n fails to fulfill the claims of counterparties, payments are made proportional to liabilities. In particular, if both commercial banks have insurance and both are owed p', n makes the maximum payment it can and divides it equally between the commercial banks.

Equilibrium. We make several parametric assumptions. Each commercial bank can repay its depositors if its project succeeds without liquidating any of its illiquid assets: $r_c > 1 + p$. Commercial banks are willing to invest in their projects using deposits: $\sigma_c (r_c - 1) > (1 - \sigma_c)\lambda_c$. If no projects fail, there are no liquidations: $pr_n > p' > 1 + p$. Finally, we focus on $1 + p > l_n$. This means that if n has low return from its investment and its counterparties have positive claims, then n must liquidate all of its illiquid assets to in trying to fulfill the claims of its counterparties. In other words, the contracts are not fully securitized and insurance involves counterparty risk.

Denote

$$
\begin{aligned}
A &= \frac{\sigma_n \left(r_n p - (1 - \sigma_c)\, p'\right)}{(1 - \sigma_n)(1 - \sigma_c)\lambda_n} \\
B &= 1 + p - \frac{1}{1 - \sigma_n} - \frac{C}{\lambda_c} \\
C &= \frac{\sigma_n}{1 - \sigma_n}\left(p' - 1 - p\right) - \frac{\sigma_c}{1 - \sigma_c}\frac{p}{1 - \sigma_n}
\end{aligned}
$$

We report results on $l_n < A$ and relegate $l_n > A$ to the appendix. The specification $l_n < A$ implies that the investment bank's expected losses are small relative to the expected gains from its project so that the investment bank prefers to sell as much insurance as possible.

Proposition 9. *If $l_n < B$, there is no insurance. If $B < l_n < \frac{2}{1+\sigma_c}B$, only one commercial bank has insurance. If $\frac{2}{1+\sigma_c}B < l_n$, both commercial banks have insurance.*

If $l_n < B$, an insurance contract is not individually rational for commercial banks as the investment bank does not have sufficient collateral. If $l_n > B$, there is sufficient collateral. Commercial banks find it individually rational to have insurance. However, if $l_n < \frac{2}{1+\sigma_c}B$, commercial banks refrain from having insurance at the same time, since the investment bank does not have sufficient collateral to cover for the the event that both projects of commercial banks fail. If $l_n > \frac{2}{1+\sigma_c}B$, the investment bank has sufficient collateral to make insurance sufficiently secure for both commercial banks at the same time.

The relevant case for network hazard is $B < l_n < \frac{2}{1+\sigma_c}B$. This is when each commercial banks find insurance individually rational, but only if the other commercial bank does not buy insurance. If the investment bank's project fails, it must rely on its collateral to make payments to commercial banks. Its collateral is limited, and must be shared between all claimant commercial banks whose projects have failed. Each commercial bank understands that if the other commercial bank also buys insurance, and its project fails, there will be less to claim from the investment bank, which deters the commercial bank from buying insurance. One commercial bank buys insurance and the other does not.

Welfare and interventions. Suppose that there is a government that can intervene with capital injections after project returns materialize before liquidations take place. We call an optimal capital injection that makes the an institution not liquidate any illiquid assets a bailout provided that the institution would liquidate some illiquid assets otherwise. Bailouts are funded by distortionary taxation and each unit of transfer has cost $\kappa \geq 1$. The cost of bailouts is an important determinant of optimal interventions. To highlight network hazard in the most clear way we assume $\kappa < \lambda_c$.

Proposition 10. *Under $\kappa < \lambda_n$, the investment bank is bailed out when it faces liquidations. Accordingly, commercial banks never need bailouts. The network formed formed involves two insurance contracts.*

Under $\kappa > \lambda_n$, only commercial banks are bailed out when they face liquidations. If $l_n < -C$, there is no insurance. If $0 < -C < l_n$, there is one insurance contract. If $0 < C$, there are two insurance contracts.

In the case of $\kappa < \lambda_n$, the investment bank is bailed out when it faces liquidations. Then there is no need to bailout an insured commercial bank as its counterparty risk is eliminated by the bailout of the investment bank, and the insurance contract prevents liquidation by the commercial bank when the investment bank's project succeeds. Under $\kappa > \lambda_n$, only the commercial banks receive bailouts. One may think that it could be optimal to bailout the investment bank as these transfers would be channeled to commercial banks, which would reduce the cost of bailing out the commercial banks. This not the case. Capital injections, either directly to commercial banks or through the investment bank, aims to prevent the inefficient liquidation of illiquid assets. The critical observation is that the financial system as a whole must liquidate the amount that just suffices to pay the depositors. Preventing liquidations that would suffice to pay the depositors is cheaper if commercial banks are bailed out directly because $\lambda_c > \lambda_n$.

Under $\kappa < \lambda_n$, the investment bank is saved from any liquidation, and so it always makes full payments to a claimant commercial bank. A commercial bank whose project fails receives p' but pays $1 + p$ to its depositors, and retains $p' - 1 - p$. This is moral hazard since the counterparty risk that the commercial bank has taken is indirectly rewarded by the bailout of the investment bank at expense of taxpayers.

Banks benefit from bailouts they receive as it saves them from elastic liquidation costs. Moral hazard on the intensive margin conceals the network hazard on the extensive margin. Network hazard is illuminated the most under $\kappa > \lambda_n$ in the case of $B < l_n < \frac{2}{1+\sigma_c}B$ and $0 < C$. In this case, in the absence of interventions, it is already individually rational for a commercial bank to buy insurance, but this is prevented by the second-order counterparty risk. If c_2 has insurance, c_1 faces second-order counterparty risk should it buy insurance. Bank c_2 can fail, which then increases the direct counterparty exposure of c_1 to n, while such an exposure would be profitable for c_1 if c_2 had not bought insurance. This is distinct from typical forms of moral hazard. For example, in the case of $l_n < B$ and $0 < l_n + C$, c_1 would not find it individually rational to buy insurance in the absence of interventions, but interventions incentivize c_1 to buy insurance regardless of whether c_2 buys insurance or not. This is standard moral hazard. Network hazard is distinct in that it reduces second-order counterparty risk, and the sole mitigation of contagion leads to a more interconnected network.

The next task is then to figure out the welfare and stability implications of such an interconnected network induced by the presence of interventions. We maintain $\kappa > \lambda_n$, $B < l_n < \frac{2}{1+\sigma_c}B$, and $0 < C$ to single out the effects of network hazard as much as possible. The distribution of welfare is reported in the appendix. To discipline the costs and simplify expressions, we take $\lambda_c, \lambda_n, \kappa$ to be close to each other: $\lambda_n \gtrsim \kappa \gtrsim \lambda_c$. Denote $R_n = r_n p$, $\kappa' = (1+p)\kappa$, $\kappa'' = (1-\sigma_c)(1+p)\kappa$. Note that $\kappa'' > \kappa$.[11]

Proposition 11. *The difference in expectation and variance of ex-post welfare between the presence and absence of interventions is given by*

$$\begin{aligned}
\Delta\mathbb{E}[w] =& \sigma_n R_n - (1-\sigma_c)(\kappa''-\kappa)\\
\Delta\, Var[w] =& \sigma_n(1-\sigma_n)\left[4\left(R_n + 2\sigma_c r_c - \kappa''\right)^2 - \left(R_n^2 + R_n\kappa' + \kappa''^2\right)\right]\\
&+ (1-\sigma_n)\sigma_c(1-\sigma_c)\left[2\left(r_c+\kappa'\right)^2 - \left(2r_c^2 + r_c\kappa' + \kappa'^2\right)\right]\\
&- \sigma_c(1-\sigma_c)\left[\left(r_c+\kappa\right)^2 - r_c^2\right]
\end{aligned}$$

If $R_n > \kappa''$ there exists r_c^* such that for all $r_c > r_c^*$, $\Delta\, Var[w] > 0$.

[11] By $B < l_n < \frac{2}{1+\sigma_c}B$ we have $B > 0$. Then by $C > 0$ we have $\kappa'' > \kappa$.

Welfare is typically increased in expectation. As R_n can be arbitrarily large, the additional investment by the investment bank covers for the added cost of bailouts and liquidations. Regarding the variance of welfare, the terms in $\Delta\mathrm{Var}[w]$ are instructive. The term with $+\sigma_n(1-\sigma_n)$ captures mainly the additional variance born out of the increased investment by the investment bank. The term with $+(1-\sigma_n)\,\sigma_c(1-\sigma_c)$ is the network hazard term. By the fact that both commercial banks now buy insurance, they are both exposed to a bad shock to the investment bank. This creates additional variance proportional to the individual variance of commercial banks' payoffs, amplified by the failure probability of the investment bank. The term with $-\sigma_c(1-\sigma_c)$ is the removal of the variance from the commercial bank that would not buy insurance in the absence of interventions. When $R_n > \kappa''$, the term with $\sigma_n(1-\sigma_n)$ dominates the term with $-\sigma_c(1-\sigma_c)$. Then the network hazard term with $+(1-\sigma_n)\,\sigma_c(1-\sigma_c)$ makes variance larger in the presence of interventions. The idiosyncratic risk of the investment bank becomes a source of aggregate volatility due to network hazard.

6 Conclusion

Contagion poses a substantial concern in prominent markets. Authorities' interventions in contagion through protective measures, such as vaccines, bailouts, and subsidies, can inadvertently reduce endogenous market discipline against contagion, leading to networks that are more susceptible to it. As links in networks are frequently formed by mutual consent, this outcome represents a form of collective moral hazard, referred to as network hazard. One particularly noteworthy and nuanced aspect is that agents become less concerned about their connections *transmitting* contagion, rather than merely originating it. This dynamic is not present in standard economic models that do not incorporate granular networked interactions and contagion. The interplay between protective measures and network hazard may result in decreased welfare in response to improved protective measures, despite complete information. A more concentrated network and consequently higher variance in welfare and contagion reach are common outcomes. Larger variance holds economic significance in the context of discontinuous social costs arising from limited hospital capacity during pandemics, political backlash against large-scale bailout plans, or any convex ex-post cost of contagion and interventions. Important future research directions include examining additional and detailed manifestations of network hazard in each of the environments demonstrated in this study.

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A Proofs

Notation. Throughout the appendix we denote $x = p_1 \circ x_1 \oplus p_2 \circ x_2 \oplus ...$ a random variable that takes value x_i with probability p_i for each i. Also $\mathbb{B}[k, x]$ is the binomial distribution with k tries and x success probability. Welfare in the absence of interventions is denoted w whereas welfare in the presence of interventions is denoted w'.

A.1 Coordination games and interventions

(Proposition 1) Safe agents do not form a link between each other. Then there are three possible networks; no links, one link, two links; between r and the safe agents. As b_i is arbitrarily large, any agent with a bad shock chooses $a_i = 0$. Also, if s_k has a link with r and r chooses 0, then s_k chooses 0 because $g_s > 0$.

Under $\beta_r < 2g_r$, if r has two links and at least one s chooses 0, then r chooses 0 since $\beta_r < 2g_r$. This implies that in equilibrium, in a connected component with at least one link, all agents choose 0 if at least one agent has a bad shock, and all agents choose 1 if no agent has a bad shock. Then r's expected payoff as a function of the number of its links d is $U_{r,d} = \alpha_r \alpha_s^d d \left(\beta_r - g_r\right)$. By $2\alpha_s > 1$ and $\beta_r > g_r$, $U_{r,d}$ is increasing in $d \leq 2$. The payoff of s_k if both s_1, s_2 are connected to r is $U_{s,2} = \alpha_r \alpha_s^2 \left(\beta_s - g_s\right) - c$. If one is connected and the other is not, the one that is connected has $U_{s,1} = \alpha_r \alpha_s \left(\beta_s - g_s\right) - c$. Notice $U_{s,1} > U_{s,2}$. Then if $U_{s,2} > 0$ both safe agents form links with r. If $U_{s,1} > 0 > U_{s,2}$, then one forms a link and the other does not. If $U_{s,1} < 0$ there are no links.

Under $\beta_r > 2g_r$, r chooses 0 if and only if r gets a bad shock or all of its connections get bad shocks. In either case, all agents choose 0. If r gets a good shock, each s chooses with its own shocks: 0 if and only if the shock is bad. Then r's expected payoff as a function of the number of its links is $U_{r,1} = \alpha_r \alpha_s \left(\beta_r - g_r\right)$ or

$$\begin{aligned} U_{r,2} &= \alpha_r \left(\alpha_s^2 2 \left(\beta_r - g_r\right) + 2\alpha_S(1-\alpha_S)\left(\beta_r - 2g_r\right)\right) \\ &> \alpha_r \alpha_s^2 2 \left(\beta_r - g_r\right) > \alpha_r \alpha_s \left(\beta_r - g_r\right) = U_{r,1} \end{aligned}$$

by $2\alpha_s > 1$. So r prefers to have more links. When s_k gets a good shock, r chooses 0 only when r gets a bad shock. So the payoff of s_k is $U_{s,1} = \alpha_s \alpha_r \left(\beta_s - g_s\right) - c$ if it has a link and 0 otherwise. If $U_{s,1} > 0$, i.e. $\kappa < \alpha_r \alpha_s$ both s form links with r. Otherwise there are no links.

(Proposition 2) Consider the auxiliary problem of choosing an action profile a to maximize $V = \sum_i \left(a_i \sum_j \left(\beta_{ij} a_j - \tau_i\right) e_{ij}\right)$

Given that b_i is large enough, $a_i^* = 0$ if $\tau_i = b_i$. Given this,

$$\begin{aligned} V &= \sum_{i:\tau_i = g_i} a_i \left(\left(\sum_{j \neq i:\tau_j = g_j} \beta_{ij} a_j e_{ij} \right) - d_i g_i \right) \\ &= \sum_{i:\tau_i = g_i} \sum_{j \neq i:\tau_j = g_j} a_i a_j \beta_{ij} e_{ij} - \sum_{i:\tau_i = g_i} a_i d_i g_i \end{aligned}$$

If i has no neighbors, i's action is efficient. So there are no transfers.

If i has neighbors but all of i's neighbors have bad shocks, then $a_i^* = 0$ to save on $\sum_{i:\tau_i = g_i} a_i d_i g_i$ even if i has a good shock.

If i has a good shock and it has a neighbor with a good shock, say j, then there are two cases. If the third agent also has a good shock, there is no need for transfers; all agents choose 1. If the third agent is has a bad shock there are two cases. If the third agents is not connected to i or j, then i and j do not need transfers and they choose 1. So the only case there can possibly be an optimal and positive transfer is when all agents are connected, r and one s_k have good shocks, and other $s_{k'}$ has a bad shock. Due to the complementarities, it is either optimal that r and s_k both choose 0 or they both choose 1. If they both choose 0, $V = 0$. If they both choose 1, $V = V^* \equiv \beta_r + \beta_s - 2g_r - g_s$.

This shows that if $V^* < 0$, then $W \leq 0$. Then choosing $t = 0$ implements that optimal action profile. In this case there are never any transfers and the network formed is the same with the absence of interventions.

If $V^* > 0$, optimal action profile is implemented by

$$\begin{aligned} t_r(1) &= (2g_r - \beta_r)^+ , \;\; t_r(0) = 0 \\ t_{s_k}(1) &= (g_s - \beta_s)^+ = 0, \;\; t_{s_k}(0) = 0 \\ t_{s_{k'}} &= 0 \end{aligned}$$

If $2g_r < \beta_r$, there is no need for transfers: $t = 0$ and r chooses 1. If $2g_r > \beta_r$, then an s agent has expected payoff $U_{s,1} = \alpha_r \alpha_s (\beta_s - g_s) - c$ regardless of whether the other s agents has a link with r or not. r, conditional on degree d, has expected payoff $U_{r,d} = \alpha_r \alpha_s^d d\left(\beta_r - g_r\right)$ which is increasing in d. Thus the unique stable network has two links if $\alpha_r\alpha_s > \kappa$ and 0 links if $\alpha_r\alpha_s < \kappa$.

(Proposition 3) Denote $v_s = \beta_s - g_s$, $v_r = \beta_r - g_r$.

$$\begin{aligned}
w + c &= \left(\alpha_r\alpha_s \circ \left(v_s + v_r\right) + 0\right) \\
\mathbb{E}[w] &= \alpha_r\alpha_s\left(v_s + v_r\right) - c \\
\mathrm{Var}[w] &= \alpha_r\alpha_s\left(1 - \alpha_r\alpha_s\right)\left(v_s + v_r\right)^2
\end{aligned}$$

Some algebra yields that in the presence of interventions

$$\begin{aligned}
& w' + 2c = \alpha_s^2\alpha_r \circ \left(2v_s + 2v_r\right) \oplus 2(1-\alpha_s)\alpha_s\alpha_r \circ \left(v_r - \beta_r + v_s\right) \\
& \oplus (1-\alpha_s)^2\alpha_r \circ 0 \oplus \left(1 - \alpha_r\right) \circ 0 \\
& \mathbb{E}[w'] = 2\alpha_s\alpha_r\left(v_s + v_r - (1-\alpha_s)\beta_r\right) - 2c \\
& \mathrm{Var}[w'] = 2\alpha_r\alpha_s\left(1 - \alpha_s\right)\Big[\left(v_s + v_r\right)^2 \\
& + \left(1 - 2(1-\alpha_s)\alpha_s\right)\beta_r + 2\left(2\alpha_s - 1\right)\beta_r\left(v_r + v_s\right)\Big] \\
& + 4\alpha_r\left(1 - \alpha_r\right)\alpha_s^2\left(v_s + v_r - (1-\alpha_s)\beta_r\right)^2
\end{aligned}$$

Then by rearranging terms we get

$$\begin{aligned}
& \mathrm{Var}[w'] - \mathrm{Var}[w] = \left(1 + 2\alpha_s - 3\alpha_s\alpha_r\right)\left(v_s + v_r\right)^2 + \\
& + 2\left(1 - \alpha_s\right)\left(1 - 2\alpha_r\alpha_s\left(1 - \alpha_s\right)\right)\beta_r - 4\left(1 - \alpha_s\right)\left(1 - 2\alpha_r\alpha_s\right)\left(v_r + v_s\right)\beta_r \\
& > 0 \impliedby \left(1 + 2\alpha_s - 3\alpha_s\alpha_r\right)\left(1 - 2\alpha_r\alpha_s\left(1 - \alpha_s\right)\right) > 2\left(1 - \alpha_s\right)\left(1 - 2\alpha_r\alpha_s\right)^2
\end{aligned}$$

Denote $x = \alpha_r\alpha_s$ and $y = 1 - \alpha_s$. Then

$$\begin{aligned}
& \mathrm{Var}[w'] - \mathrm{Var}[w] > 0 \impliedby \\
& Q[x;y] \equiv -\left(2y\right)x^2 + x\left(-3 + 4y^2 + 2y\right) + \left(3 - 4y\right) > 0
\end{aligned}$$

Q is a concave quadratic in x. The end points for x are given by $x = \alpha_r\alpha_s \in [0, \alpha_s^2] = [0, (1-y)^2]$. Given that $y = 1 - \alpha_s < 0.5$, at both end points $x = 0$ and $x = (1-y)^2$, $Q[0;y]$ and $Q[(1-y)^2; y]$ are positive. So $Q[x;y]$ is positive.

The difference in mean is

$$\mathbb{E}[w'] - \mathbb{E}[w] = \alpha_s\alpha_r\left(v_s + v_r - 2(1-\alpha_s)\beta_r\right) - c$$

which can be positive or negative. Pick any $\alpha_s > 0.5$, any $\alpha_r < \alpha_s$, any $\beta_s > v_s > \beta_r - 2v_r > 0$, and let $c = \alpha_s\alpha_r\left(v_s + v_r - 2(1-\alpha_s)\beta_r\right) - x$. (Note that $\mathbb{E}[w'] - \mathbb{E}[w] = x$) This clearly implies all parametric conditions except $\alpha_r\alpha_s > \frac{c}{v_s} > \alpha_r\alpha_s^2$. Note

$$\begin{aligned}
\alpha_r\alpha_s &> \frac{\alpha_s\alpha_r\left(v_s + v_r - 2(1-\alpha_s)\beta_r\right) - x}{v_s} > \alpha_r\alpha_s^2 \\
\iff 0 &< 2\beta_r - \frac{v_r}{(1-\alpha_s)} + \frac{x}{(1-\alpha_s)} < v_s
\end{aligned}$$

Then by assuming

$$0 < 2\beta_r - \frac{v_r}{(1-\alpha_s)} < v_s$$

x can have positive or negative sign. For example, $\alpha_s > 0.5$, $\alpha_r < \alpha_s$, $\beta_s > v_s > \beta_r > 0$ and $v_r < \min\left\{\frac{1}{2}, 2\left(1-\alpha_s\right)\right\}\beta_r$ satisfies all conditions.

A.2 Epidemics and protective measures

(Proposition 4) The payoff to a b-type from having no connection is $V_0 = -\kappa\eta$. The payoff to a b-type from being connected to a_j if a_j has no other connection is $V_0 + V_j$ where $V_j = v_j - 1 - \kappa(1-\eta)\tau\eta$. The payoff to a b-type from being connected to a_j if a_j has one more connection is $V_0 + V_j - \Delta$ where $\Delta = \kappa(1-\eta)\tau(1-\eta)\eta\tau$.

It is easy to see that the stability is characterized as follows. $V_1 < 0$, there are no links. If $V_1 > 0 > \max\{V_2, V_1 - \Delta\}$, then one connected to a_1, the other no connections. If $V_2 = \max\{V_2, V_1 - \Delta\} > 0$, then one to a_1 one to a_2. If $V_1 - \Delta = \max\{V_2, V_1 - \Delta\} > 0$, then both connected to a_1. Regarding m, these bounds correspond to

$$V_i < 0 \iff m_i^* \equiv \frac{v_i - 1}{\kappa(1-\eta)\eta\tau_0} < m$$
$$V_1 - \Delta < 0 \iff m^* \equiv \frac{\sqrt{1 + 4\frac{v_1 - 1}{\kappa\eta}} - 1}{2(1-\eta)\tau_0} < m$$
$$V_1 - \Delta < V_2 \iff m^{**} \equiv \sqrt{\frac{v_1 - v_2}{\kappa\eta(1-\eta)^2\tau_0^2}} < m$$

Then the conditions are: If $m > m_1^*$, there are no links. If $m_1^* > m > \max\{m_2^*, m^*\}$, then one connected to a_1, the other no connections. If $m_2^* > m > m^{**}$, then one to a_1 one to a_2. If $\min\{m^{**}, m^*\} > m$, then both connected to a_1. Note

$$m_2^* > m > m^* \iff V_2 > 0 > V_1 - \Delta \implies V_2 > V_1 - \Delta \iff m > m^{**}$$

meaning $m_2^* > m^* \implies m^* > m^{**}$ by picking $m = m^* + \epsilon$. Also,

$$m_2^* < m < m^* \iff V_2 < 0 < V_1 - \Delta \implies V_2 < V_1 - \Delta \iff m < m^{**}$$

meaning $m_2^* < m^* \implies m^* < m^{**}$by picking $m = m^* - \epsilon$. So we have either $m_2^* > m^* > m^{**}$ or $m_2^* < m^* < m^{**}$.

Consider $m_2^* < m^* < m^{**}$. If $m > m_1^*$, there are no links. If $m_1^* > m > m^*$, then one connected to a_1, the other no connections. If $m^* > m$, then both connected to a_1.

Next consider $m_2^* > m^* > m^{**}$. If $m > m_1^*$, there are no links. If $m_1^* > m > m_2^*$, then one connected to a_1, the other no connections. If $m_2^* > m > m^{**}$, then one to a_1 one to a_2. If $m^{**} > m$, then both connected to a_1.

Thus by defining $m_1 \equiv m_1^*$, $m_2 \equiv \max\{m_2^*, m^*\} \leq m_3 \equiv \min\{m^{**}, m^*\}$, we complete the proof.

(Proposition 5) When both b-types are connected to the same a-type, the number of infections X, its mean, and its variance are

$$
\begin{aligned}
X &= \mathbb{B}[1,\eta] + \Big[\left(\eta^3 + \eta^2 (1-\eta)(4-\tau)\tau + 3(1-\eta)^2 \eta\tau^2\right) \circ 3 \\
&\quad \oplus \left(\eta^2 (1-\eta)(1-\tau)(3-\tau) + 4\eta (1-\eta)^2 \tau (1-\tau)\right) \circ 2 \\
&\quad \oplus \left(\eta (1-\eta)^2 (1-\tau)(3-\tau)\right) \circ 1 \oplus (1-\eta)^3 \circ 0 \Big] \\
\mathbb{E}[X] &= \eta(4 + \tau(1-\eta)(4 + (2-3\eta)\tau)) \\
\mathrm{Var}[X] &= (1-\eta)\eta\Big(12\tau(\tau+1) + 4 - \eta\tau\Big(\tau\big((2-3\eta)^2(1-\eta)\tau^2 \\
&\quad + 8(1-\eta)(2-3\eta)\tau + 45 - 34\eta\big) + 16\Big)\Big)
\end{aligned}
$$

When b-type agents are connected to separate a-type agents

$$
\begin{aligned}
X &= \left[\left(\eta^2 + 2\eta(1-\eta)\tau\right) \circ 2 \oplus (2\eta(1-\eta)(1-\tau)) \circ 1 \oplus (1-\eta)^2 \circ 0\right] \\
&\quad + \left[\left(\eta^2 + 2\eta(1-\eta)\tau\right) \circ 2 \oplus (2\eta(1-\eta)(1-\tau)) \circ 1 \oplus (1-\eta)^2 \circ 0\right] \\
\mathbb{E}[X] &= 4\eta\left(1 + (1-\eta)\tau\right) \\
\mathrm{Var}[X] &= 4\eta(1-\eta)\left(1 + (3-4\eta)\tau + 2\eta(1-\eta)\tau^2\right)
\end{aligned}
$$

When only one b-type agent is connected to an a-type agent, and the others have no connections,

$$
\begin{aligned}
&X = \left[\left(\eta^2 + 2\eta(1-\eta)\tau\right) \circ 2 \oplus (2\eta(1-\eta)(1-\tau)) \circ 1 \oplus (1-\eta)^2 \circ 0\right] + \mathbb{B}[2,\eta] \\
&\mathbb{E}[X] = 2\eta\left(2 + (1-\eta)\tau\right) \\
&\mathrm{Var}[X] = 2\eta(1-\eta)\left(2 + (3-4\eta)\tau + 2\eta(1-\eta)\tau^2\right)
\end{aligned}
$$

When there are no connections, the X is $\mathbb{B}[4,\eta]$. The expectation is 4η. The variance is $4\eta(1-\eta)$.

Recall the proof of Proposition 4. Focus on the case of $m_2^* > m^* > m^{**}$. As m goes down, at $m = m_1 = m_1^*$, the network switches from empty to having one link. Expectation and variance clearly increase. At $m = m_2 = m_2^*$, the network switches from one connection to two separate connections. Then expectation and variance change by

$$\begin{aligned}
\Delta\mathbb{E}[X] &= [4\,(\eta + (1-\eta)\tau\eta)] - [2\,(2\eta + (1-\eta)\tau\eta)] = 2(1-\eta)\tau\eta > 0 \\
\Delta\mathrm{Var}[X] &= \left[4(1-\eta)\left(\eta + (3-4\eta)\,\tau\eta + 2(1-\eta)\tau^2\eta^2\right)\right] \\
&\quad - \left[2(1-\eta)\left(2\eta + (3-4\eta)\,\tau\eta + 2(1-\eta)\tau^2\eta^2\right)\right] \\
&= 2(1-\eta)\left((3-4\eta)\,\tau\eta + 2(1-\eta)\tau^2\eta^2\right) > 0
\end{aligned}$$

At $m = m_3 = m^{**}$ expectation and variance change by

$$\begin{aligned}
&\Delta\mathbb{E}[X] = \eta(4 + \tau(1-\eta)(4 + (2-3\eta)\tau)) - 4\left(\eta + (1-\eta)\tau^2\eta\right) \\
&= (1-\eta)(2-3\eta)\tau^2\eta > 0 \\
&\Delta\mathrm{Var}[X] = (1-\eta)\eta\Bigg(12\tau(\tau+1) + 3 - \eta\tau\Bigg(\tau\big((2-3\eta)^2(1-\eta)\tau^2 \\
&+ 8(1-\eta)(2-3\eta)\tau + 45 - 34\eta\big) + 16\Bigg)\Bigg) + \eta(1-\eta) \\
&- 4\eta(1-\eta)\big(1 + (3-4\eta)\,\tau + 2\eta(1-\eta)\tau^2\big) > 0 \iff \\
&- (2-3\eta)^2\tau^4\eta^2 + 8(2-3\eta)\tau^2\eta + \frac{\eta\,(12 - 53\eta + 42\eta^2)}{1-\eta} > 0
\end{aligned}$$

where the last inequality holds by $\eta < 1/4$ and $\tau^2\eta > 0$.

Next consider $m_2^* < m^* < m^{**}$. At $m = m_1^*$, expectation and variance clearly increase. At $m_2 = m_1 = m^*$, network switches from one link to both b-type agents having a connection with a_1. Then expectation and variance change by the sum of the two ΔExp and ΔVar terms above, which are both positive. So both changes are positive.

A.3 Supply chains and subsidies

(Proposition 6) Take u and consider $D_u = \{d\}$. Conditional on good shocks and being supplied, the downstream firm has ex-post payoff $-c + p_D$ from production, so it produces if supplied. The supplier u has interim payoff

$-k + \delta p_U > 0$ from production so it produces and supplies.

Consider $D_u = D$. Conditional on good shocks and being supplied, d has ex-post payoff $-c + p_D$. The supplier u can produce 1 or 2. If it produces 1, it has interim payoff $-k + 2\delta(1-\delta)p_U$. If it produces 2, it has interim payoff $-2k + (\delta^2 + 2\delta(1-\delta))\, p_U$. Then it produces 1 if and only if $k > \delta^2 p_U$.

Then under $k < \delta^2 p_U$, both downstream firms choose u_1 as $\alpha_{u_1} > \alpha_{u_2}$. Under $k > \delta^2 p_U$, if both downstream firms choose u_1 they each have ex-ante payoff $\alpha_{u_1}\delta\left(1-\delta+\frac{\delta}{2}\right)(-c+p_D)$. If they choose separate suppliers, the one with smaller payoff has ex-ante payoff $\alpha_{u_2}\delta\,(-c+p_D)$. Then they choose separate suppliers if and only if $\frac{\alpha_{u_2}}{\alpha_{u_1}} > 1 - \delta + \frac{\delta}{2}$.

(Proposition 7) The subsidies are injected into the firms, which are than paid to other non-modeled agents in the economy who produce the inputs at cost $e_{u/d}$. The subsidies are then transfer from taxpayers to producers of external inputs. Given $e_d = 0$ and $e_u = k_u$, welfare is given by $\sum_{d\in D} pq_d - \sum_{u\in U} k_u q_u$. Since $k' > p$, w is maximized by $q_d = 1$ if $k_{u_d} = k$ and $q_d = 0$ otherwise. The minimal subsidies that implement this outcome is $s_d = c' - p_D$ if $c_d = c'$, which induces d to produce, and all other subsidies are 0. Then an upstream firm u with two downstream buyers and a good shock has payoff $q_u\,(-k + p_U)$ from producing q_u, so it produces 2. This means both downstream are supplied conditional on their supplier getting a good shock so they both choose u_1 as $\alpha_{u_1} > \alpha_{u_2}$.

(Proposition 8) In the absence of interventions,

$$
\begin{aligned}
w &= (\alpha_{u_1}\delta \circ (p-k) \oplus \alpha_{u_1}(1-\delta) \circ (-k) \oplus (1-\alpha_{u_1}) \circ 0) \\
&\quad + (\alpha_{u_2}\delta \circ (p-k) \oplus \alpha_{u_2}(1-\delta) \circ (-k) \oplus (1-\alpha_{u_2}) \circ 0) \\
\mathbb{E}[w] &= (\alpha_{u_1} + \alpha_{u_2})\,(\delta p - k) \\
\mathrm{Var}[w] &= \alpha_{u_1}\delta\,(1-\alpha_{u_1}\delta)\,p^2 + \alpha_{u_1}(1-\alpha_{u_1})k^2 - 2\delta\alpha_{u_1}(1-\alpha_{u_1})pk \\
&\quad + \alpha_{u_2}\delta\,(1-\alpha_{u_2}\delta)\,p^2 + \alpha_{u_2}(1-\alpha_{u_2})k^2 - 2\delta\alpha_{u_2}(1-\alpha_{u_2})pk
\end{aligned}
$$

In the presence of interventions,

$$\begin{aligned}
w' &= \alpha_{u_1} \circ 2(p-k) \oplus (1-\alpha_{u_1}) \circ 0 \\
\mathbb{E}[w'] &= 2\alpha_{u_1}(p-k) \\
\mathrm{Var}[w'] &= \alpha_{u_1}(1-\alpha_{u_1})\, 4(p-k)^2
\end{aligned}$$

Clearly $\mathbb{E}[w'] > \mathbb{E}[w]$. Let $\alpha \approx \alpha_{u_i}$. Then

$$\begin{aligned}
&\mathrm{Var}[w'] > \mathrm{Var}[w] \iff \alpha_{u_1}(1-\alpha_{u_1})\, 4(p-k)^2 > \\
&\alpha_{u_1}\delta\,(1-\alpha_{u_1}\delta)\, p^2 + \alpha_{u_1}(1-\alpha_{u_1})k^2 - 2\delta\alpha_{u_1}(1-\alpha_{u_1})pk \\
&+ \alpha_{u_2}\delta\,(1-\alpha_{u_2}\delta)\, p^2 + \alpha_{u_2}(1-\alpha_{u_2})k^2 - 2\delta\alpha_{u_2}(1-\alpha_{u_2})pk \\
&\iff \left(2 - \delta\frac{1-\alpha\delta}{1-\alpha}\right) + \left(\frac{k}{p}\right)^2 - (4-2\delta)\left(\frac{k}{p}\right) > 0 \impliedby 1 - \frac{\delta}{2\,(1-\delta)} > \alpha
\end{aligned}$$

A.4 Derivatives and bailouts

(Proposition 9) We first find the expected payoffs for possible network structures. Remember $l_c > 1 + p$ meaning that the commercial banks' illiquid assets never bind the possible payments to depositors.

If c_k does not invest, it has payoff $V_{c,\emptyset} \equiv \lambda_c l_c$. If it invests and does not buy insurance, it has payoff $V_{c,0} \equiv \lambda_c l_c + \sigma_c(r_c - 1) - (1-\sigma_c)\lambda_c$. If n does not sell insurance it has payoff $V_{n,0} \equiv \lambda_n l_n$.

If only one commercial bank buys insurance, say c_k, it has payoff

$$\begin{aligned}
&V_{c,1} \equiv \lambda_c l_c + \sigma_c(r_c - 1 - p) \\
&+ (1-\sigma_c)\,\sigma_n\left[(\min\{p', pr_n + l_n\} - 1 - p)^+ - (1 + p - \min\{p', pr_n + l_n\})^+\,\lambda_c\right] \\
&+ (1-\sigma_c)(1-\sigma_n)\left[\min\{p', l_n\} - (1 + p - \min\{p', l_n\})^+\,\lambda_c\right] \\
&= \lambda_c l_c + \sigma_c(r_c - 1 - p) + (1-\sigma_c)\,\sigma_n(p' - 1 - p) \\
&- (1-\sigma_c)(1-\sigma_n)(1 + p - l_n)\,\lambda_c
\end{aligned}$$

and n has payoff

$$
\begin{aligned}
&V_{n,1} \equiv \lambda_n l_n + \sigma_n \sigma_c r_n p \\
&+ \sigma_n (1-\sigma_c) \left[(pr_n - p')^+ - \min\left\{ (p' - pr_n)^+, l_n \right\} \lambda_n \right] \\
&+ (1-\sigma_n)\sigma_c [0] + (1-\sigma_n)(1-\sigma_c) [-\min\{p', l_n\} \lambda_n] \\
&= \lambda_n l_n + \sigma_n (r_n p - (1-\sigma_c) p') - (1-\sigma_n)(1-\sigma_c) l_n \lambda_n
\end{aligned}
$$

If both commercial banks buy insurance, they each have payoff

$$
\begin{aligned}
&V_{c,2} \equiv \lambda_c l_c + \sigma_c (r_c - 1 - p) \\
&+ (1-\sigma_c)\sigma_n\sigma_c \left[(\min\{p', 2pr_n + l_n\} - 1 - p)^+ - (1 + p - \min\{p', 2pr_n + l_n\})^+ \lambda_c \right] \\
&+ (1-\sigma_c)^2 \sigma_n \left[(\min\{p', pr_n + l_n/2\} - 1 - p)^+ - (1 + p - \min\{p', pr_n + l_n/2\})^+ \lambda_c \right] \\
&+ (1-\sigma_c)(1-\sigma_n)\sigma_c \left[(\min\{p', l_n\} - 1 - p)^+ - (1 + p - \min\{p', l_n\})^+ \lambda_c \right] \\
&+ (1-\sigma_c)(1-\sigma_n)(1-\sigma_c) \left[(\min\{p', l_n/2\} - 1 - p)^+ - (1 + p - \min\{p', l_n/2\})^+ \lambda_c \right] \\
&= \lambda_c l_c + \sigma_c (r_c - 1 - p) + (1-\sigma_c)\sigma_n (p' - 1 - p) \\
&- (1-\sigma_c)(1-\sigma_n)\sigma_c (1 + p - l_n) \lambda_c - (1-\sigma_c)^2 (1-\sigma_n) (1 + p - l_n/2) \lambda_c
\end{aligned}
$$

whereas n has payoff

$$
\begin{aligned}
&V_{n,2} \equiv \lambda_n l_n + \sigma_n \sigma_c^2 r_n p \\
&+ 2\sigma_n\sigma_c(1-\sigma_c) \left[(2pr_n - p')^+ - \min\left\{ (p' - 2pr_n)^+, l_n \right\} \lambda_n \right] \\
&+ \sigma_n (1-\sigma_c)^2 \left[2 (pr_n - p')^+ - \min\left\{ 2 (p' - pr_n)^+, l_n \right\} \lambda_n \right] \\
&+ 2 (1-\sigma_n) \sigma_c (1-\sigma_c) [-\min\{p', l_n\} \lambda_n] \\
&+ (1-\sigma_n) (1-\sigma_c)^2 [-\min\{2p', l_n\} \lambda_n] \\
&= \lambda_n l_n + 2\sigma_n (pr_n - (1-\sigma_c) p') - (1-\sigma_n) (1-\sigma_c^2) l_n \lambda_n
\end{aligned}
$$

Stable network. Recall

$$A \equiv \frac{\sigma_n \left(r_n p - (1-\sigma_c)\, p'\right)}{(1-\sigma_n)(1-\sigma_c)\lambda_n}$$

Then some algebra shows

$$V_{n,2} > V_{n,1} \iff \frac{A}{\sigma_c} > l_n$$
$$V_{n,1} > V_{n,0} \iff A > l_n$$

Recall

$$B \equiv 1 + p - \frac{(1-\sigma_c)\sigma_n \left(p' - 1 - p\right) - \sigma_c p}{(1-\sigma_n)(1-\sigma_c)\lambda_c} - \frac{1}{1-\sigma_n}$$

Then some algebra shows

$$V_{c,0} > V_{c,2} \iff \frac{2}{1+\sigma_c} B > l_n$$
$$V_{c,0} > V_{c,1} \iff B > l_n$$

Then the stable network is given by the following table:

	$l_n < B$	$B < l_n < \frac{2}{1+\sigma_c}B$	$\frac{2}{1+\sigma_c}B < l_n$
$l_n < A$	no links	one link	two links
$A < l_n < \frac{A}{\sigma_c}$	no links	one link	one link
$\frac{A}{\sigma_c} < l_n$	no links	no links	no links

(Proposition 10)

Optimal transfers. If a commercial bank c_k does not have insurance and its project fails, it receives transfer $x_c = 1$ since $\lambda_c > \kappa$. The expected payoff of c_k is then

$$V'_{c,0} \equiv \lambda_c l_c + \sigma_c \left(r_c - 1\right) - (1-\sigma_c)\left(p + x - 1 - p\right)$$
$$= \lambda_c l_c + \sigma_c \left(r_c - 1\right)$$

Consider the case of one commercial bank c_k having insurance and focus on c_k and n. Conditional on a realization of shocks, let n have $e_n \in \{r_n, 0\}$ return and c have $e_c \in \{r_c, 0\}$ return. Suppose n has $d_n \in \{0, p'\}$ debt to c and c has $d_c = 1 + p$ debt to depositors. Without any transfers, ex-post payoffs are

$$\begin{aligned} v_{n,1} &= \lambda_n l_n + (e_n - d_n)^+ - \min\left\{(d_n - e_n)^+, l_n\right\}\lambda_n \\ v_{c,1} &= \lambda_c l_c + (e_c + \min\{d_n, e_n + l_n\} - d_c)^+ \\ &\quad - (d_c - e_c - \min\{d_n, e_n + l_n\})^+ \lambda_c \end{aligned}$$

Under transfers x_n to n and x_c to c we have

$$\begin{aligned} v'_{n,1} &= \lambda_n l_n + (x_n + e_n - d_n)^+ - \min\left\{(d_n - e_n - x_n)^+, l_n\right\}\lambda_n \\ v'_{c,1} &= \lambda_c l_c + (x_c + e_c + \min\{d_n, x_n + e_n + l_n\} - d_c)^+ \\ &\quad - (d_c - x_c - e_c - \min\{d_n, x_n + e_n + l_n\})^+ \lambda_c \end{aligned}$$

Since $\lambda_c > \kappa$, optimal x_c given x_n is

$$x_c = (d_c - e_c - \min\{d_n, x_n + e_n + l_n\})^+$$

Then

$$\begin{aligned} v'_{n,1} &= \lambda_n l_n + (x_n + e_n - d_n)^+ - \min\left\{(d_n - e_n - x_n)^+, l_n\right\}\lambda_n \\ v'_{c,1} &= \lambda_c l_c + \left((d_c - e_c - \min\{d_n, x_n + e_n + l_n\})^+ + e_c + \min\{d_n, x_n + e_n + l_n\} - d_c\right)^+ \\ &= \lambda_c l_c + (e_c + \min\{d_n, x_n + e_n + l_n\} - d_c)^+ \end{aligned}$$

So welfare from the insured commercial bank and the investment bank given

x_n is

$$\begin{aligned}
w(x_n) \propto & \underbrace{(x_n + e_n - d_n)^+}_{w_1(x_n)} + \underbrace{(e_c + \min\{d_n, x_n + e_n + l_n\} - d_c)^+}_{w_2(x_n)} \\
& - \underbrace{\min\left\{(d_n - e_n - x_n)^+, l_n\right\} \lambda_n}_{w_3(x_n)} \\
& - \underbrace{x_n \kappa}_{w_4(x_n)} - \underbrace{(d_c - e_c - \min\{d_n, x_n + e_n + l_n\})^+ \kappa}_{w_5(x_n)}
\end{aligned}$$

The possible terms with x_n in this expression from each summand are $+x_n$, $+x_n$, $+\lambda_n x_n$, $-\kappa x_n$, and $-\kappa x_n$. In terms of the coefficients of x_n, if $w_1(x_n) \propto x_n$, then $w_2(x_n) \propto 0$, $w_4(x_n) \propto 0$,$w_5(x_n) \propto -\kappa x_n$. Then $w \propto -(2\kappa - 1)x_n$, decreasing in x_n. If $w_1(x_n) \propto 0$ and $w_2(x_n) \propto x_n$, then $w_3(x_n) \propto 0$ and $w_5(x_n) \propto 0$. Then $w(x_n) \propto -(\kappa - 1)x_n$, decreasing in x_n. If $w_1(x_n) \propto 0$ and $w_2(x_n) \propto 0$, then $w(x_n) \propto \{-2\kappa x_n, -(\kappa - \lambda_n)x_n\}$ is decreasing in x_n under $\kappa < \lambda_n$. Hence w is decreasing in x_n for $\kappa < \lambda_n$. So when $\kappa < \lambda_n$, optimally

$$\begin{aligned}
x_n &= 0, \\
x_c &= (d_c - e_c - \min\{d_n, e_n + l_n\})^+
\end{aligned}$$

This is, under $\kappa < \lambda_n$, c_k is bailed out directly and no transfers are made to n. Then

$$\begin{aligned}
v'_{n,1} &= \lambda_n l_n + (e_n - d_n)^+ - \min\left\{(d_n - e_n)^+, l_n\right\} \lambda_n \\
v'_{c,1} &= \lambda_c l_c + (e_c + \min\{d_n, e_n + l_n\} - d_c)^+
\end{aligned}$$

and so the expected payoffs are

$$\begin{aligned}
& V'_{n,1} = V_{n,1} \\
& V'_{c,1} = \lambda_c l_c + \sigma_c (r_c - 1 - p) + (1 - \sigma_c) \sigma_n (\min\{p', p r_n + l_n\} - 1 - p)^+ \\
& + (1 - \sigma_c)(1 - \sigma_n) \min\{p', l_n\} \\
& = \lambda_c l_c + \sigma_c (r_c - 1 - p) + (1 - \sigma_c) \sigma_n (p' - 1 - p) + (1 - \sigma_c)(1 - \sigma_n) l_n
\end{aligned}$$

Under $\lambda_n > \kappa$ it is already optimal to bailout n to save its own liquidation costs to maximize

$$v_{n,1} = (x_n + e_n - d_n)^+ - \min\left\{(d_n - e_n - x_n)^+, l_n\right\}\lambda_n$$

by setting $x_n = (d_n - e_n)^+$. So when $\lambda_n > \kappa$, optimally

$$\begin{aligned} x_n &= (d_n - e_n)^+, \\ x_c &= (d_c - e_c - d_n)^+ = 0 \end{aligned}$$

Note that in the specific parameters of the model, $d_c - e_c - d_n \leq 1 + p - p' < 0$ and so $x_c = 0$. This is, the commercial banks never need bailouts.

Then

$$\begin{aligned} v'_{n,1} &= \lambda_n l_n + \left((d_n - e_n)^+ + e_n - d_n\right)^+ = \lambda_n l_n + (e_n - d_n)^+ \\ v'_{c,1} &= \lambda_c l_c + (e_c + d_n - d_c)^+ \end{aligned}$$

Then expected payoffs are

$$\begin{aligned} V'_{n,1} &= \lambda_n l_n + \sigma_n\left(r_n p - (1 - \sigma_c)\, p'\right) \\ V'_{c,1} &= \lambda_c l_c + \sigma_c\left(r_c - 1 - p\right) + (1 - \sigma_c)\,\sigma_n\left(p' - 1 - p\right) \end{aligned}$$

Next consider the case wherein both commercial banks have insurance. Conditional on a realization of shocks, let n have $e_n \in \{r_n, 0\}$ return and c_t have $e_{ct} \in \{r_c, 0\}$ return for $t = 1, 2$. Suppose n has $d_{nt} \in \{0, p'\}$ debt to c_t and c_t has $d_c = 1 + p$ debt to depositors. Denote $d_n = d_{n1} + d_{n2}$. Then

$$\begin{aligned} v'_{n,2} &= \lambda_n l_n + (x_n + e_n - d_n)^+ - \min\left\{(d_n - e_n - x_n)^+, l_n\right\}\lambda_n \\ v'_{c,2,t} &= \lambda_c l_c + \left(x_{ct} + e_{ct} + \min\left\{d_{nt}, \frac{d_{nt}}{d_n}(x_n + e_n + l_n)\right\} - d_c\right)^+ \\ &\quad - \left(d_c - x_{ct} - \min\left\{d_{nt}, \frac{d_{nt}}{d_n}(x_n + e_n + l_n)\right\}\right)^+ \lambda_c \end{aligned}$$

Then by $\lambda_c > \kappa$ optimally

$$x_{ct} = \left(d_c - e_{ct} - d_{nt} \min\left\{1, \frac{x_n + e_n + l_n}{d_n}\right\}\right)^+$$

Then

$$\begin{aligned}
w \propto\ & (x_n + e_n - d_n)^+ + \sum_t \left(e_{ct} + d_{nt} \min\left\{1, \frac{x_n + e_n + l_n}{d_n}\right\} - d_c\right)^+ \\
& - \min\left\{(d_n - e_n - x_n)^+, l_n\right\} \lambda_n \\
& - \kappa x_n - \kappa \sum_t \left(d_c - e_{ct} - d_{nt} \min\left\{1, \frac{x_n + e_n + l_n}{d_n}\right\}\right)^+
\end{aligned}$$

The same logic as before shows that $x_n = 0$ under $\lambda_n < \kappa$. So[12]

$$x_n = 0, x_{ct} = \left(d_c - e_{ct} - d_{nt} \min\left\{1, \frac{e_n + l_n}{d_n}\right\}\right)^+$$

Then

$$\begin{aligned}
v'_{n,2} &= \lambda_n l_n + (e_n - d_n)^+ - \min\left\{(d_n - e_n)^+, l_n\right\} \lambda_n = v_{n,2} \\
v'_{c,2,t} &= \lambda_c l_c + \left(e_{ct} + d_{nt} \min\left\{1, \frac{e_n + l_n}{d_n}\right\} - d_c\right)^+
\end{aligned}$$

[12]If $\lambda_n > \kappa$, then n would be bailed out as before, and commercial banks would receive payments that just suffice:

$$\begin{aligned}
x_n &= (d_n - e_n)^+ \\
x_{ct} &= \left(d_c - e_{ct} - d_{nct} \min\left\{1, \frac{(d_n - e_n)^+ + e_n + l_n}{d_n}\right\}\right)^+
\end{aligned}$$

This makes the expected payoffs

$$
\begin{aligned}
V'_{n,2} &= V_{n,2} \\
V'_{c,2} &= \lambda_c l_c + \sigma_c \left(r_c - 1 - p\right) \\
&\quad + (1-\sigma_c)\sigma_n\sigma_c \left(\min\left\{p', 2pr_n + l_n\right\} - 1 - p\right)^+ \\
&\quad + (1-\sigma_c)\sigma_n(1-\sigma_c)\left(\min\left\{p', pr_n + \frac{l_n}{2}\right\} - 1 - p\right)^+ \\
&\quad + (1-\sigma_c)(1-\sigma_n)\sigma_c \left(\min\left\{p', l_n\right\} - 1 - p\right)^+ \\
&\quad + (1-\sigma_c)(1-\sigma_n)(1-\sigma_c)\left(\min\left\{p', \frac{l_n}{2}\right\} - 1 - p\right)^+ \\
&= \lambda_c l_c + \sigma_c \left(r_c - 1 - p\right) + (1-\sigma_c)\sigma_n \left(p' - 1 - p\right)
\end{aligned}
$$

When $\lambda_n > \kappa$, as before, n is bailed out for its own sake and so

$$
\begin{aligned}
x_n &= \left(d_n - e_n\right)^+, \\
x_{ct} &= \left(d_c - e_{ct} - d_{nt}\right)^+ = 0
\end{aligned}
$$

Note that $d_c - e_{ct} - d_{nt} \leq 1 + p - p' < 0$ so $x_{ct} = 0$. This is, commercial banks never need bailouts. Then

$$
\begin{aligned}
v'_{n,2} &= \lambda_n l_n + \left(e_n - d_n\right)^+ \\
v'_{c,2,t} &= \lambda_c l_c + \left(e_{ct} + d_{nt} - d_c\right)
\end{aligned}
$$

Then the expected payoffs are

$$
\begin{aligned}
V'_{n,2} &= \lambda_n l_n + 2\sigma_n \left(r_n p - (1-\sigma_c)\, p'\right) \\
V'_{c,2} &= \lambda_c l_c + \sigma_c \left(r_c - 1 - p\right) + \left(1-\sigma_c\right)\sigma_n \left(p' - 1 - p\right)
\end{aligned}
$$

Stable network. Combining the cases, under $\lambda_n > \kappa$

$$\begin{aligned}
V'_{n,0} &= \lambda_n l_n \\
V'_{c,0} &= \lambda_c l_c + \sigma_c \left(r_c - 1\right) \\
V'_{n,1} &= V_{n,1} = \lambda_n l_n + \sigma_n \left(r_n p - \left(1 - \sigma_c\right) p'\right) - (1 - \sigma_n)(1 - \sigma_c) l_n \lambda_n \\
V'_{c,1} &= \lambda_c l_c + \sigma_c \left(r_c - 1 - p\right) + \left(1 - \sigma_c\right) \sigma_n \left(p' - 1 - p\right) + \left(1 - \sigma_c\right)\left(1 - \sigma_n\right) l_n \\
V'_{n,2} &= V_{n,2} = \lambda_n l_n + 2\sigma_n \left(p r_n - (1 - \sigma_c) p'\right) - \left(1 - \sigma_n\right)\left(1 - \sigma_c^2\right) l_n \lambda_n \\
V'_{c,2} &= \lambda_c l_c + \sigma_c \left(r_c - 1 - p\right) + (1 - \sigma_c)\sigma_n \left(p' - 1 - p\right)
\end{aligned}$$

Denote

$$C = \frac{(1 - \sigma_c)\sigma_n \left(p' - 1 - p\right) - \sigma_c p}{\left(1 - \sigma_c\right)\left(1 - \sigma_n\right)}$$

Some algebra shows $V'_{c,1} > V'_{c,2}$ and $V'_{c,0} > V'_{c,2} \iff 0 > C$ and $V'_{c,0} > V'_{c,1} \iff 0 > C + l_n$. Then the network formed is described by

	$l_n < -C$	$0 < -C < l_n$	$0 < C$
$l_n < A$	no links	one link	two links
$A < l_n < \frac{A}{\sigma_c}$	no links	one link	one link
$\frac{A}{\sigma_c} < l_n$	no links	no links	no links

Under $\lambda_n < \kappa$,

$$\begin{aligned}
V'_{n,0} &= \lambda_n l_n \\
V'_{c,0} &= \lambda_c l_c + \sigma_c \left(r_c - 1\right) \\
V_{n,1} &= \lambda_n l_n + \sigma_n \left(r_n p - \left(1 - \sigma_c\right) p'\right) \\
V_{c,1} &= \lambda_c l_c + \sigma_c \left(r_c - 1 - p\right) + \left(1 - \sigma_c\right) \sigma_n \left(p' - 1 - p\right) \\
V'_{n,2} &= \lambda_n l_n + 2\sigma_n \left(r_n p - \left(1 - \sigma_c\right) p'\right) \\
V'_{c,2} &= \lambda_c l_c + \sigma_c \left(r_c - 1 - p\right) + \left(1 - \sigma_c\right) \sigma_n \left(p' - 1 - p\right)
\end{aligned}$$

Clearly $V'_{c,2} = V'_{c,1} > V'_{c,0}$ and $V'_{n,2} > V'_{n,1} > V'_{n,0}$. So the network formed involves two insurance contracts.

(Proposition 11) In the absence of interventions

$$w \propto \left[\sigma_c \sigma_n \circ (R_n + r_c) \oplus \sigma_c (1-\sigma_n) \circ (r_c) \oplus (1-\sigma_c)\, \sigma_n \circ (R_n) \right.$$
$$\left. \oplus\, (1-\sigma_c)\,(1-\sigma_n) \circ (-\lambda_n l_n - \lambda_c\,(1+p-l_n)) \right]$$
$$+ \left[\sigma_c \circ (r_c) \oplus (1-\sigma_c) \circ (-\lambda_c)\right]$$

In the presence of interventions

$$\begin{aligned}
w' &\propto \sigma_c^2 \sigma_n \circ (2R_n + 2r_c) \\
&\oplus 2\sigma_c\,(1-\sigma_c)\,\sigma_n \circ (2R_n + r_c) \oplus (1-\sigma_c)^2\,\sigma_n \circ (2R_n) \\
&\oplus \sigma_c^2\,(1-\sigma_n) \circ (2r_c) \oplus 2\sigma_c\,(1-\sigma_c)\,(1-\sigma_n) \circ (r_c - \lambda_n l_n - \kappa\,(1+p-l_n)) \\
&\oplus (1-\sigma_c)^2\,(1-\sigma_n) \circ (-\lambda_n l_n - 2\kappa\,(1+p-l_n/2))
\end{aligned}$$

By $\lambda_n \gtrsim \kappa \gtrsim \lambda_c$

$$w \propto \left[\sigma_c \sigma_n \circ (R_n + r_c) \oplus \sigma_c (1-\sigma_n) \circ (r_c) \oplus (1-\sigma_c)\, \sigma_n \circ (R_n) \right.$$
$$\left. \oplus\, (1-\sigma_c)\,(1-\sigma_n) \circ (-\,(1+p)\,\kappa) \right]$$
$$+ \left[\sigma_c \circ (r_c) \oplus (1-\sigma_c) \circ (-\kappa)\right]$$

and

$$\begin{aligned}
w' &\propto \sigma_c^2 \sigma_n \circ (2R_n + 2r_c) \\
&\oplus 2\sigma_c\,(1-\sigma_c)\,\sigma_n \circ (2R_n + r_c) \oplus (1-\sigma_c)^2\,\sigma_n \circ (2R_n) \\
&\oplus \sigma_c^2\,(1-\sigma_n) \circ (2r_c) \oplus 2\sigma_c\,(1-\sigma_c)\,(1-\sigma_n) \circ (r_c - \kappa\,(1+p)) \\
&\oplus (1-\sigma_c)^2\,(1-\sigma_n) \circ (-2\kappa\,(1+p))
\end{aligned}$$

Then

$$\begin{aligned}
\mathbb{E}[w] &= \sigma_n R_n + 2\sigma_c r_c - (1-\sigma_n)(1-\sigma_c)(1+p)\kappa - (1-\sigma_c)\kappa \\
\mathbb{E}[w'] &= \sigma_n 2R_n + 2\sigma_c r_c - (1-\sigma_n)\,2\,(1-\sigma_c)(1+p)\kappa \\
\mathbb{E}[w'] - \mathbb{E}[w] &= \sigma_n R_n - (1-\sigma_c)\left[(1-\sigma_n)(1+p) - 1\right]\kappa \\
&= \sigma_n R_n - (1-\sigma_c)(\kappa'' - \kappa)
\end{aligned}$$

By the conditional variance formula

$$\begin{aligned}
&Var[w] = \sigma_c(1-\sigma_c)(r_c+\kappa)^2 \\
&+ \sigma_n(1-\sigma_n)R_n^2 + \sigma_c(1-\sigma_c)r_c^2 \\
&+ (1-\sigma_c)(1-\sigma_n)(1-(1-\sigma_c)(1-\sigma_n))((1+p)\kappa)^2 \\
&+ (\sigma_n R_n + \sigma_c r_c)(1-\sigma_c)(1-\sigma_n)(1+p)\kappa
\end{aligned}$$

$$\begin{aligned}
&Var[w'] = \sigma_n\left[2\sigma_c(1-\sigma_c)r_c^2\right] + (1-\sigma_n)\left[2\sigma_c(1-\sigma_c)(r_c+\kappa(1+p))^2\right] \\
&\sigma_n(1-\sigma_n)\left[(2R_n + 2\sigma_c r_c) + (2\sigma_c r_c - 2(1-\sigma_c)\kappa(1+p))\right]
\end{aligned}$$

Then by rearranging terms we have

$$\begin{aligned}
&Var[w'] - Var[w] = \sigma_n(1-\sigma_n)\left[4(R_n + 2\sigma_c r_c - \kappa'')^2 - \left(R_n^2 + R_n\kappa' + \kappa''^2\right)\right] \\
&+ (1-\sigma_n)\sigma_c(1-\sigma_c)\left[2(r_c+\kappa')^2 - \left(2r_c^2 + r_c\kappa' + \kappa'^2\right)\right] - \sigma_c(1-\sigma_c)\left[(r_c+\kappa)^2 - r_c^2\right] \\
&= \sigma_c(1-\sigma_c)r_c\left[16\sigma_n(1-\sigma_n)(R_n - \kappa'')\sigma_c + 3\kappa'' - 2\kappa + o(\frac{1}{r_c})\right]
\end{aligned}$$

By $B < l_n < \frac{2}{1+\sigma_c}B$, $B > 0$. Then by $C > 0$, $(1+p)(1-\sigma_n) > 1$. Then $\kappa'' > 1$ so $3\kappa'' - 2\kappa > 0$. Then if $R_n > \kappa''$, $Var[w'] - Var[w] > 0$ for large r_c.