Relationship Externalities

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Abstract

We study a model of network externalities transmitted over links wherein the externalities stem from links themselves rather than nodes. For example, bilateral investments can fail and trigger cascading defaults of firms. Joint research projects can succeed and innovations can be diffused over the network. We characterize stable and efficient networks. Under negative externalities, disjoint cliques are stable and efficient. Under positive externalities complete networks and star networks are stable. Efficient networks feature are a mix: pineapple networks which consist of one large clique and a star network appended to each other.

JEL classification: D62, D85

Keywords: Network formation, Stability, Strong Stability, Externalities

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1 Introduction

Networks of relationships and the externalities they generate shape the flow of resources and information. Therefore, it is important to understand which networks will form and whether they are efficient. To do so we propose a general model of network formation where an agent's payoff depends on own degree *and* the number of links in the component it belongs to. Formally, the utility of an agent will be denoted $U(d, e)$ where d is its degree in the network and e is the number of links in its component. This formulation allows us to capture two kinds of externalities in reduced form.

U increasing in d but decreasing in e captures negative link-externalities. Links incident to oneself are beneficial but links incident to others are not. For example, in co-authorship, the coauthors of coauthors create negative externalities because of the competition for attention. More seriously, our formulation allows one to encode in reduced form the effect of contagion. Existing contagion models, for example Kempe et al. (2003) among others, focus on contagion that starts with nodes which become intractable once one tries to encode the fine details of the contagion process. Further, in these models links serve only as a conduit for externalities. In our formulation, the links themselves can trigger a contagion. For example, links could be joint investments, which, when they fail initiate contagion.

U decreasing in d but increasing in e captures positive link-externalities. Incident links are costly, but other's links are beneficial. For example, in an R&D network, if two companies collaborate on a project, it increases the chances of a discovery and positive spillovers (see Goyal and Moraga-Gonzalez (2001), König et al. (2012)).

Our main contribution is a characterization of strongly stable networks under both negative and positive link externalities. Our characterization does not rely on a particular functional form for $U(d, e)$ which will make it useful for applications. In the presence of negative link-externalities, strongly stable networks consist of disjoint cliques of a particular size. Simply put, we show that in a deviation, the worst off deviator cannot get better off. The deviator with smallest degree, say d, cannot get better off because other deviators have larger degree and so the total link-externalities are exceed $d(d+1)/2$. This cannot be better than the optimal clique. In this sense, the optimally sized cliques make sure that potential deviators have conflict among them. This argument fails for cliques of different sizes from the particular or for a structure other than cliques. Under positive link-externalities, we find that strongly stable networks are star networks or complete networks. The center of the star network serves as a public good provider and enjoys significantly less payoff than the other agents. In all of these cases, strongly stable networks are Pareto efficient, but not necessarily efficient. We identify various instances of when strong stability implies efficiency.

These results hold even with heterogenous payoff functions, i.e., the payoff to agent i is $U_i(d, e, f)$ where f is the number of nodes in the component of node i, U_i is strictly decreasing/increasing in e_i, weakly decreasing/increasing in f_i, and depends arbitrarily on d_i.

The next section introduces the model and notation. Section 3 discusses the case of negative externalities while Section 4 focuses on positive externalities. Section 5 discusses efficiency and examples while Section 6 describes some extensions of our model. A discussion of the prior literature can be found in Section 7.

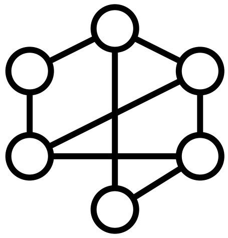
2 Model

Preliminaries A *network* G is a pair (N, E) where $N = \{1, 2, ..., n\}$ is a finite set of *nodes* and $E \subset [N]^2$ is a set of *links*.[1] Throughout this paper we assume $n \geq 2$. Call two nodes $i, j \in N$ *neighbors* in G if $\{i, j\} \in E$. A link $\{i, j\} \in E$ is *incident* to i and j in G. For $i \in N$, denote by $N_{i,G} = \{j \in N_G | \{i, j\} \in E\}$ the set of neighbors of i in G and call $d_{i,G} = |N_{i,G}|$ the *degree* of i in G. A *path* from i_1 to i_k in G is a sequence of distinct nodes $(i_1, i_2, ..., i_k)$, all in N, such that $\{i_t, i_{t+1}\} \in E$ for all t. Two nodes i and j are *connected* in G if there is path in G between them. Network $G' = (N_{G'}, E_{G'})$ is a *subnetwork* of network G if $N_{G'} \subset N$ and $E_{G'} \subset E \cap [N_{G'}]^2$ denoted $G' \preceq G$. Note that G is a subnetwork of itself. A subnetwork $G' \preceq G$ is *connected* if every pair of nodes in $N_{G'}$ is connected in G' or if $|N_{G'}| = 1$. The *component* of a node i in G is the largest connected subnetwork of G that includes the node i with respect to the order $\preceq$. Denote $e_{i,G}$ the number of links in the component of i in G. The *payoff* to node i from network G is $u_{i,G} = U(d_{i,G}, e_{i,G})$ where $U : \mathbb{Z}^2 \to \mathbb{R}$.

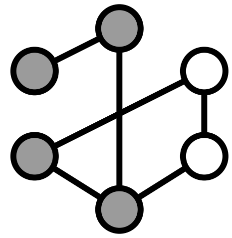
Stability Consider a candidate network $G = (N, E)$ and a subset of nodes $D \subset N$. A feasible deviation from G by D allows nodes in D to *simultaneously* add any set of absent links within D and delete any set of links incident to any node in D. Formally, network $G' = (N, E') \neq G$ is a *feasible deviation* from G by $D \subset N$ if $E' \cap [N \backslash D]^2 = E \cap [N \backslash D]^2$ and $E' \cap ([N]^2 \backslash [D]^2) \subset E \cap ([N]^2 \backslash [D]^2)$. A feasible deviation is illustrated in Figure 1. A *profitable deviation* from G by D is a feasible deviation from G by D that strictly improves the payoff of each member of D: $u_{i,G'} > u_{i,G}$ for all $i \in D$. G is called $k-$*stable* if there is no profitable deviation from G by any $D \subset N$ with $|D| \leq k$. We refer to 2-stable networks as *stable networks*, and $\infty-$stable networks as *strongly stable networks*.

Other definitions that will be used throughout follow next. We place them in one place for the convenience of the reader. For each integer k, let $B(k) = \frac{1}{2}k(k-1)$. A *connected network* is

[1] $[X]^2$ is the mathematical notation for the set of all 2-element subsets of a set X.

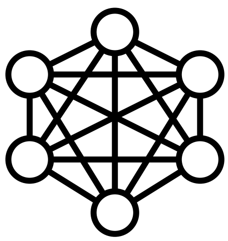


Candidate network G

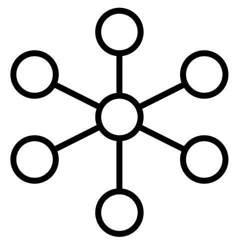


A feasible deviation by the grey nodes.

Figure 1: A feasible deviation



A clique



A star

Figure 2: A clique and a star

a network that is a connected subnetwork of itself. Call a subnetwork $G' \preceq G$ *the subnetwork induced by* $N_{G'}$ if $E_{G'} = E \cap [N_{G'}]^2$. Notice that components are induced subnetworks. The *order* of a subnetwork is the number of nodes in it. A *clique* is a subnetwork in which all nodes are neighbors of each other. A *disjoint clique* is a clique which is also a component. A $k-$clique is disjoint clique of order k. A *complete network* is a network which is a clique. A *star* is a component in which one node called the *center* is neighbors with all other nodes called *leaves* wherein no two leaves are neighbors of each other. A *star network* is one where the network itself is a star component. A clique and a star are illustrated in Figure 2. A component is called *non-trivial* if there are at least two links in it, a *link-component* if there is exactly one link in it, and a *node-component* if there are no links in it. A *cycle* in G is a sequence of nodes $(i_1, i_2, ..., i_k)$ where $\{i_t, i_{t+1}\} \in E$ for all t and it holds that $i_t = i_{t'}$ iff $\{t, t'\} = \{1, k\}$. A node i is called a *strongly connected node (SC)* if it belongs to a cycle, and a *weakly connected node (WC)* otherwise. Call a node i *central (C)* if $d_i = n - 1$ and *non-central (NC)* if $d_i \neq n - 1$.

3 Negative externalities

Suppose U is weakly increasing in d and strictly decreasing in e, namely *negative link-externalities.*

Proposition 1. *Under negative link-externalities, any stable network consists of disjoint cliques.*

The proof is simple. If there are two nodes i and j who are not neighbors, but share a common neighbor k, then i and j can each cut their link with k and add the link $\{i, j\}$. This decreases the number of links in the component of i and j, leaving the degrees of i and j unchanged. Therefore, the payoffs of i and j both strictly increase. As simple as it is, this idea is insightful. One can increase the degrees of nodes in $D \subset N$ by either adding links inside D or by adding links between D and $N \backslash D$. Conditional on the desired increment to the degrees of nodes in D, the latter option entails twice the amount of negative externalities than the former option. High clustering minimizes the negative externalities imposed on a set of nodes conditional their degrees.

There can be multiple stable networks and clique size is not pinned down uniquely by stability. See Section 5 for an example. As a selection, we use strong stability. Denote the expected payoff of a node in a $(d+1)$-clique $V(d) = U(d, B(d+1))$. Let $d^* = argmax_d V(d)$. We assume that d^* is well defined, which is generically true.

Theorem 1. *Suppose that* n *is divisible by* d^*+1*. Under negative link-externalities, a network is strongly stable iff it consists of* (d^*+1)*-cliques.*

For example consider $U(d, e) = d - ce$ where $c > 0$. Then $V(d) = d - \frac{c}{2}d(d+1)$. Take $\frac{1}{c} + \frac{1}{2} \in \mathbb{N}$ for simplicity. Then, a strongly stable network consists of disjoint $(\frac{1}{c} + \frac{1}{2})$-cliques. We explore this example in more detail in Section 5.

The proof uses a method we call the min-cut deviation. Suppose a profitable deviation exists. Among all such deviations, choose one with the fewest number of links between the deviators and the non-deviators after the deviation. We show that the deviator who has the the smallest degree after this deviation, say d', cannot achieve a payoff more than $V(d')$, which is weakly less than $V(d^*)$.

As Jackson and Van den Nouweland (2005) point out, strong stability is a solution concept that is rarely satisfied. A situation wherein strongly stable networks exist is when all nodes can jointly achieve the maximal payoff they could achieve across all networks, as in Erol (2019). In our framework, nodes may not desire more links but they are strictly harmed by their neighbors having more links. That is, each node would strictly prefer to be at the center of a star network of appropriate order. But this is impossible. Thus, the incentives of nodes are misaligned and they cannot simultaneously achieve maximal payoffs. This conflict between nodes in terms of who gets a higher degree militates against the existence of strongly stable networks. Interestingly, this same force implies the deviators face the same conflict among themselves. The worst-off deviator gets exposed to high negative externalities by the better-off deviators. Therefore, under negative link-externalities, notions of farsightedness are not required to discipline the plausibility of deviations. It is remarkable that strongly stable networks exist in our framework because negative externalities are proportional to the number of edges whereas the benefits of links are proportional to degrees, and the former grows exponentially compared to the latter.

4 Positive externalities

Next we study the case of U that is weakly decreasing in d and strictly increasing in e, namely *positive link-externalities.*

Proposition 2. *Under positive link-externalities, a stable network consists of either one non-trivial component and some node-components, or some link-components and some node-components. Call the former non-trivial stable networks and the latter trivial stable networks.*

Roughly stated, the main idea is that under positive externalities separate non-trivial components can be combined to boost link-externalities. This is not the case if the cost of links are too high compared to the positive externalities so that nodes want to belong to trivially small components. It can lead to trivial stable networks. We provide a characterization of trivial stable networks in Proposition 7 in the Appendix. We focus on non-trivial stable networks.

To obtain a sharper characterizations we impose more structure on U. If $U(d,e) < U(d+1,e+1)$ for all $d \leq e$, then, all pairs would add all the missing links. The unique stable network is the complete network. If $U(d,e) < U(d-1,e-1)$ for all $d \leq e$, the unique stable network is the empty network. To see why, recall that a node is called strongly connected if it belongs to a cycle. Every non-trivial component must have a strongly connected node in it (see Lemma 1 in the Appendix). But, a strongly connected node i can cut a single link and increase its payoff to $U\left(d_i-1, e_i-1\right)$ with a unilateral deviation. Link-components also benefit from cutting their links. Such cases of global monotonicity yield uninteresting results. The problem is more interesting when there are non-trivial tradeoffs. Accordingly, we assume that $U(d,e)$ is *strictly quasi-convex*: $U(d,e) < \max\{U(d-1,e-1), U(d+1,e+1)\}$ for all $1 \leq d \leq e$. Roughly, for any node, it is either profitable to add a link or cut a link. However, the profitable option between cutting a link or adding a link varies between nodes.

Theorem 2. *Under positive link-externalities and strict quasi-convexity, non-trivial stable networks are either star networks or complete networks.*

A star network is stable iff $U(1,n-1) \geq U(2,n)$ *and* $U(n-1,n-1) \geq U(0,0)$. *A stable star network is strongly stable iff* $U(1,n-1) \geq U(n-1,B(n))$.

A complete network is stable iff $U(n-1,B(n)) \geq \max\{U(1,1+B(n-1)), U(0,0)\}$. *A stable complete network is strongly stable.*[2]

We illustrate with some examples. Consider the additive form $U(d,e) = e - cd$ where $c < 1$. It is strictly quasi-convex. In this case, the unique stable network is the complete network and it is also strongly stable.

Next, consider the multiplicative form $U(d,e) = \frac{e}{d+\epsilon}$ where $1 > \epsilon > 0$. It is strictly quasi-convex and the unique stable network is the star network. Moreover, $U(1,n-1) \geq U(n-1,B(n))$. Thus, the star network is strongly stable.

Strongly stable star networks are interesting. In our framework, the center of the star is providing a public good by connecting all nodes to each other. The leaves enjoy higher payoffs than the center. Recall that in the case of negative link-externalities, all nodes would strictly prefer to be at the center of a star component, but this posed some conflict among agents. Under positive link-externalities, nodes strictly prefer to be a leaf of the star network, yet one node still rises to be the center to increase the overall connectivity of the network, providing a public good.

5 Examples, Stability, and Efficiency

Here we revisit the examples provided earlier and identify stable, strongly stable, and efficient networks. We start with the case of negative externalities.

Proposition 3. *Take* $U(d,e) = d - ce$ *for some* $c > 0$. *Under the appropriate parity assumptions, the following hold. A network is stable if and only if it consists of disjoint cliques with orders* $k_1 \leq k_2 \leq ... \leq k_s$ *where* $k_s \leq \frac{2}{c} + 1$ *and* $k_2 \geq \frac{1}{2}(\sqrt{\frac{8}{c} - 7} - 1)$. *A network is strongly stable if and only if it consists of disjoint* $(\frac{1}{c} + \frac{1}{2})$*-cliques. A network is efficient if and only if it consists of disjoint* $(\frac{1}{c} + \frac{1}{2})$*-cliques*

[2] In fact, a stable complete network is strongly stable for any U with positive externalities even if U is not quasi-convex.

By Theorem 1, the strongly stable network is given by disjoint $(\frac{1}{c}+\frac{1}{2})$-cliques (under the appropriate divisibility assumption). All stable networks must consist of disjoint cliques according to Proposition 1. The conditions to preclude profitable deviations by pairs reduces to $k_s \leq \frac{2}{c}+1$ and $k_2 \geq \frac{1}{2}(\sqrt{\frac{8}{c}-7}-1)$. So, in general, cliques in a stable network can be twice as large or *exponentially* smaller than the strongly stable cliques. This illustrates the weakness of stability. Nodes can get "stuck" at small cliques. When only pairs are allowed to add links in a feasible deviation, nodes from separate cliques can enjoy the benefit of having one extra link, but they expose themselves to large negative externalities due to existing links in the other node's existing clique. In contrast, strong stability allows for joint deviations. Two separate cliques can form all of their missing links and form a new and larger clique. Each deviator is still exposed to large negative externalities due to the pre-existing links in the other clique, but the benefit is larger since many links are being added at once for each deviator. As for efficiency, observe that, for a given $k < \frac{2}{c}$, it is efficient to have all links in the component. Also observe that it is not efficient to have any components with order larger than $\frac{2}{c}$. So the efficient network must consist of cliques. Then efficiency requires maximizing the average payoff which coincides with strong stability.

In general, efficiency and strong stability don't coincide. Strong stability implies Pareto efficiency, but not efficiency. This can be seen in a simple example. Suppose that $n=4$ and U satisfies

$$3U(3,4)+U(1,4) > U(1,1) > U(0,0), U(2,3), U(3,6)$$

This is consistent with negative link-externalities. The unique strongly stable network is given by two links by Theorem 1. Yet, the network that consists of a triangle with the extra node attached to it, i.e. $E=\{\{1,2\},\{2,3\},\{3,1\},\{1,4\}\}$, gives higher total payoff than the strongly stable network. Nevertheless, it can be shown that if $U(d,e)$ is additive in d, as is the case in the example, strong stability and efficiency roughly coincide.

Proposition 4. *Consider negative link-externalities. Suppose that n is divisible by d^*+1 and $U(d,e)$ is additive in d. Further, assume $U(2x,x)$ is an increasing in x and $U(0,0)>0$. Then, efficient networks and strongly stable networks coincide.*

$U(0,0)>0$ can be seen as a normalization. If we drop the assumption that $U(2x,x)$ is increasing, then strongly stable networks are still "roughly" efficient. Each component must still be close to a clique in the sense that a component with k nodes can not have less than $B(k-1)+1$ links in the efficient network. Then, the optimal number of nodes in each component is close to d^* it U is sufficiently "smooth."

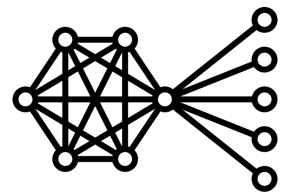


Figure 3: A pineapple graph, $P(11, 6)$

Next we consider examples for positive link-externalities.

Proposition 5. *Take $U(d, e) = e - cd$ where $c > 0$. For $c > \frac{n}{2}$, the unique stable network, the unique strongly stable network, and the unique efficient network is the empty network. For $\frac{n}{2} > c > 1$, the unique stable network is the empty network. The unique efficient network is the complete network. There are no strongly stable networks. For $c < 1$, the unique stable network, the unique strongly stable network, and the unique efficient network is the complete network.*

Next consider $U(d, e) = e - cd$. This is a case of positive externalities. For $c < 1$ we have already determined that the unique non-trivial stable and strongly stable network is the complete network. Clearly, this is also the efficient network. For $1 < c$, the unique stable network is the empty network. But this is neither strongly stable nor efficient unless $c > \frac{n-2}{2}$ which is bound to fail for large networks.

Next, consider a more interesting case of positive externalities. A pineapple graph $P(n, k)$ is found by appending $n - k$ nodes to one single node in clique of order k. Figure 3 illustrates a pineapple graph.

Proposition 6. *Take $U(d, e) = \frac{e}{d+\epsilon}$ for some arbitrarily small $\epsilon > 0$. Then, the star network is the unique strongly stable network, and the unique non-trivial stable network. The unique efficient network is given by $P(n, k_n)$ where $k_n \approx \frac{2n}{3}$.*

By Theorem 2, the unique non-trivial stable network is the star network, which is also strongly stable. For efficiency, note that in a given component $G' \preceq G$ with nodes N', the sum of payoffs is $\frac{1}{2}(\sum_{i \in N'} d_i)(\sum_{i \in N'} \frac{1}{d_i})$. The term $\sum_{i \in N'} \frac{1}{d_i}$ is the inverse degree of the graph. This has been an object of interest in Chemical Graph Theory and we make use of an existing result from Hu et al. (2007) in the proof of efficiency.

6 Extensions

There are various ways in which our results can be extended. For brevity we content ourselves with a summary here. Theorems and proofs are available upon request.

One can account for payoffs that feature node externalities and non-monotonicity in d. Denote by f_i the number of nodes in the component of i and suppose that the payoff from a network is $U(d_i, e_i, f_i)$. If U is strictly decreasing in e_i and weakly decreasing in f_i with no restriction on how it depends on d_i, all of our results in the negative-externalities section go through. If U is strictly increasing in e_i and weakly weakly in f_i with no assumption on how it depends on d_i, all of our results in the positive-externalities section go through, except Theorem 2. In that case, there can be stable networks that consist of one disjoint clique and some node-components. This happens because of the weakness of $2-$stability in that it does not allow larger coalitions of nodes to add links at the same. We can still identify the exact conditions under which such extra stable networks exist. Note that results on strongly stable networks remain unaffected.

We can also introduce heterogeneity into the payoff functions: $U_i(d_i, e_i, f_i)$. For the case of negative node/link-externalities, all results can be replicated to account for heterogenous payoff functions U_i under appropriate parity assumptions in the spirit of $\frac{n}{d^*+1} \in \mathbb{Z}$. For the case of positive node/link-externalities, all results go through if we additionally assume that $U_i(2, 3, 4) > U_i(1, 1, 2)$ for all $i \in N$.

7 Related literature

A general way to model payoffs is to use a payoff function $U_i(G)$ that depends on the entire network. This encompasses our framework and there are a large number of papers on stability under such a general framework (see sections 8.2.1. and 8.2.2. in Vannetelbosch and Mauleon (2015))Naturally this leads to weaker results. These papers provide characterizations only under weaker notions of stability or explicit functional forms for the payoff functions which focus on node-externalities. We characterize the strongly stable networks under qualitative assumptions on the payoff functions that emphasie link-externalities.

Our negative externality case is roughly related to models of contagion. Blume et al. (2013) finds that there exists pairwise Nash stable networks that consist of cliques in an independent cascades model. Erol (2019) shows that the clique structure is uniquely strongly stable under threshold contagion but his model features node-externalities rather than link-externalities. Regarding our positive externality results, Galeotti et al. (2006), and Hojman and Szeidl (2008) find star networks as Nash equilibria using the non-cooperative connections model of Bala and Goyal (2000). The connections model does not map on to notions of stability we examine. The closest to ours is Jackson and Wolinsky (1996) who find that star networks are pairwise stable under a specific payoff function that features positive node-externalities that decay with distance. To the best of our knowledge, there are no characterization results under link-externalities even for weaker solution concepts than strongly stable.

Regarding stability, perhaps the most relevant papers are Dutta and Mutuswami (1997) and Jackson and Van den Nouweland (2005). Our notion of strong stability is the same with the former's. The latter is stronger in that it requires immunity against Pareto improving deviations. We call this Pareto Strongly Stable (PSS). Jackson and Van den Nouweland (2005) show by example that existence of strongly stable networks does not imply the existence of PSS networks. Both notions can be appropriate for separate applications. If small side payments are possible among deviators, PSS is perhaps more suitable. If there are small costs to deviations, string stability is perhaps more suitable.

There are other reasons why Jackson and Van den Nouweland (2005) does not imply *any* of our results. Most importantly they find that in a PSS, nodes in a component must have equal payoff (Theorem 1) a consequence of immunity against *Pareto* deviations. Moreover, this theorem does not imply that a PSS with respect to an allocation rule Y is PSS with respect to an egalitarian allocation rule Y^{CE} which distributes the value of a component equally between all nodes. Their Theorem 1 does not imply the reverse either. However, their other results are solely about Y^{CE}, meaning that, by definition, all nodes in a component have the same payoff in any network, PSS or not. This is not how individual payoffs are defined in our model. It is far from trivial to obtain existence in our framework when there is no payoff sharing arrangement between nodes prior to or after deviations. On the other hand, the focus of Dutta and Mutuswami (1997) is to find value functions and sharing rules such that efficient and strongly stable networks exist. This endeavor is categorically different from that of characterizing strongly stable networks for given payoff functions.

8 Conclusion

In the study of network externalities, links are often modeled to be only a transmitter of externalities. In various scenarios, the links themselves create externalities as well as transmit these externalities. For example, coauthors of coauthors create negative externalities because time is limited. Joint projects can fail and lead to cascading defaults of real or financial firms. R&D projects lead to discoveries and innovations can diffuse over the rest of the network. We study a simple form of this type of externalities and characterize strongly stable networks. Negative externalities, such as contagion, lead to highly clustered network, in particular disjoint cliques. Positive externalities lead to either complete networks or star networks depending on the costs and benefits of links. When a star network is stable, the center of the star is a public good provider, transmitting positive externalities across the network although it is costly for the center to maintain a large number of links. Efficient networks tradeoff these two forces: a pineapple network emerges wherein a central nodes is still public goof provider being connected all other nodes, but there is also a large clique that further increases the extent of positive externalities which also benefits the center node.

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9 Proofs

Proof of Proposition 1

Suppose, for a contradiction, there exists a stable network in which there are two neighbors i and j in the same component. As i and j are connected, each has at least one neighbor. Delete one link incident to i and another incident to j, and insert the link $\{i, j\}$. The degrees of i and j are unchanged but the number of links in their component strictly decreases. Hence, this is a profitable deviation by i and j. Contradiction.

Proof of Theorem 1

Step 1. Consider any SS network G. By Proposition 1 G consists of disjoint cliques. The payoff to $i \in N$ is $V(d_i)$. If there exists a a disjoint clique with order $d + 1 > d^* + 1$ in the realized network, $d^* + 1$ members would deviate by forming a disjoint $(d^* + 1)$-clique and get $V(d^*) > V(d)$. If there exists various cliques with order less than $d^* + 1$ but the sum of their orders is larger than equal to $d^* + 1$, $d^* + 1$ many nodes in these cliques would form a disjoint $d^* + 1$ clique. Combining the two, by the parity assumption that $d^* + 1$ divides n, the unique candidate for a strongly stable network is one that consists of disjoint $(d^* + 1)$-cliques. Our next task is to show that G is SS.

Step 2. For any profitable deviation G' by N', define $\chi(G')$ as the number of links between N' and N/N' in G'. Let the minimum of χ be attained at G^*. Denote N^* the set of deviators in G^*, $E^* = E_{G^*}$, $N_1^* = \{i \in N^* : \exists j \notin N^* \, s.t. \, \{i, j\} \in E^*\}$, and $N_0^* = N^*/N_1^*$. Take the/a node $i^* \in N^*$ such that $d_{i^*,G^*} = \min\{d_{i,G^*} | i \in N^*\}$. Denote $d_* = d_{i^*,G^*}$. Denote $a = |N_{i^*,G^*} \cap (N \backslash N^*)|$ and $b = |N_{i^*,G^*} \cap N_0^*|$. Then $|N_{i^*,G^*} \cap N_1^*| = d_* - a - b$.

Step 3. Suppose that there exists $i \in N_0^*$ such that $d_i = d_*$. By definition of N_0^*, $N_{i,G^*} \subset N^*$. By definition of i^*, all $i \in N^*$ have $d_{i,G^*} \geq d_*$. Therefore, $e_{i,G^*} \geq d_i + \frac{1}{2} d_i d_* = B(d_* + 1)$. Thus $u_{i,G^*} = U(d_*, e_{i,G^*}) \leq U(d_*, B(d_* + 1)) = V(d_*) \leq V(d^*)$ which is a contradiction. Therefore, $d_i \geq d_* + 1$ for all $i \in N_0^*$. In particular, $i^* \in N_1^*$.

Step 4. Consider any $i, j \in N_1^*$ that are connected in G^*. If $\{i, j\} \notin E^*$, consider the deviation G' by N^* such that $E_{G'} \backslash E^* = \{i, j\}$ and $E^* \backslash E_{G'} = \{\{i, i'\}, \{j, j'\}\}$ for some $i', j' \notin N^*$. Notice that G' is a profitable deviation since G^* is whereas $\chi(G') < \chi(G^*)$. This is a contradiction so $\{i, j\} \in E^*$. Therefore, all nodes in N_1^* that are connected in G^* are neighbors of each other, and so make up a clique (not disjoint clique). In particular, there are at least $B(d_* - a - b + 1)$ links in within N_1^* in the component of i^* in G^*.

Step 5. Take any $i \in N_0$ that is connected to i^* in G^*. By Step 3, $d_i \geq d_* + 1$. By Step 4, $N_{i,G^*} \cap N_1^* \subset \{i^*\} \cup N_{i^*,G^*}$. Thus $|N_{i,G^*} \cap N_1^*| \leq d_* - a - b + 1$. Then $|N_{i,G^*} \cap (N \backslash N_1^*)| \geq a + b$. By definition of N_0^*, $N_{i,G^*} \cap (N \backslash N^*) = \emptyset$ and so $|N_{i,G^*} \cap N_0^*| \geq a + b$.

Accordingly, if $b \geq 1$, then $N_0 \neq \emptyset$, so there exists $j \in N_0$ such that $|N_{j,G^*} \cap N_0^*| \geq a + b$. Notice that all nodes in $N_{j,G^*} \cap N_0^*$ are connected to i^* meaning that they all have at least $a + b$ neighbors in N_0^*. Therefore, there are at least $\frac{1}{2}(a + b + 1)(a + b) = B(a + b + 1)$ links within in N_0^* that are in the component of i^* in G^*.

Step 6. Take any $i \in N_{i^*,G^*} \cap N_1^*$. By definition of i^*, $d_{i,G^*} \geq d_*$. By Step 4, $N_{i,G^*} \cap N_1^* \subset \{i^*\} \cup N_{i^*,G^*} \backslash \{i\}$. Thus, $|N_{i,G^*} \cap N_1^*| \leq d_* - a - b$. Then $|N_{i,G^*} \cap (N \backslash N_1^*)| \geq a + b$. This holds for all $i \in N_{i^*,G^*} \cap N_1^*$ and $|N_{i^*,G^*} \cap N_1^*| = d_* - a - b$. Thus the number of links between N_1^* and $N \backslash N_1^*$ in the component of i^* in G^* is at least $(d_* - a - b + 1)(a + b)$.

Step 7. By feasibility of the deviation G^*, links between N^* and N/N^* in G^* exist in G as well. Then $N_{i^*,G^*} \cap (N \backslash N^*) \subset N_{i^*,G} \cap (N \backslash N^*)$. Since G consists of disjoint cliques, all nodes in $N_{i^*,G} \cap (N \backslash N^*)$ are neighbors of each other in G. Then all nodes in $N_{i^*,G^*} \cap (N \backslash N^*)$ are neighbors of each other in G. Since G^* is a deviation by N^*, by the feasibility of the deviation, all nodes in $N_{i^*,G^*} \cap (N \backslash N^*)$ are neighbors of each other in G^*. Therefore, there are at least $B(a)$ many links within N/N^* in the component of i^* in G^*.

Combining all steps we find $e_{i^*,G^*} \geq B(d_* - a - b + 1) + \mathbb{1}_{b \geq 1} B(a + b + 1) + (d_* - a - b + 1)(a + b) + B(a) \geq B(d_* + 1)$. Accordingly, $u_{i,G^*} \leq U(d_*, B(d_* + 1)) = V(d_*) \leq V(d^*)$ which is a contradiction. There are no profitable deviations from G.

Proof of Proposition 2

Lemma 1. *In any network, every non-link component has at least one SC node. A non-link component has only one SC node iff it is a star component.*

Proof. Notice that WC nodes can not belong to any cycle. Suppose that all nodes are WC. Then the network can not have any cycles. Then the network is a tree. Then take any leaf of the tree and consider the parent of this leaf. The parent can cut the link with the leaf and the size of its component falls by 1. So the parent is an SC. Contradiction. So every network has at least one SC node.

It is clear that a star network has only one SC. Consider any network that has exactly one SC. If there exists a cycle, then all nodes in the cycle are SCs, which is at least three nodes. So there can not be a cycle. Then the network is a tree. Since all parents of all leaves are SCs, there can be only one such parent. The only tree in which there is only one common parent to all leaves is the star network. □

Step 1. Suppose that there exists two non-trivial components. By Lemma 1, both of these components have at least one SC node. Pick one form each, say i and j. Let i and j cut one link each with their component, and add the link $\{i, j\}$. This will keep their degree the same. e_i becomes $e_i - 1 + 1 + (e_j - 1) \geq e_i + 1$. Same for j. Thus this is a profitable deviation for i and j. This means there can not exist two non-trivial components.

Step 2. Denote $e^* = \max\{e \in \mathbb{Z} : U(1,1) \geq U(2,e)\}$. Take a non-trivial component with e links. Suppose that $e' < e^*$. Take one SC from the component, say i. Note that SCs have degree at least 2. Cut all links of i but one. After this unilateral deviation i's payoff is at least $U(1,1)$ whereas $U(1,1) \geq U(2,e^*) > U(2,e') \geq U(d_i,e')$. So this is a profitable deviation. So we must have $e' \geq e^*$.

Step 3. Suppose that there exists a link-component and a non-trivial component. Then consider the following deviation. Take one SC from the non-trivial component, say i, and one node from a link-component, say j. Cut one link of i and form a link between i and j. Node i's payoff becomes $U(d_i, e'+1) > U(d_i, e') \geq U(d_i, e^*)$ by Step 2. Node j's payoff becomes $U(2, e'+1) > U(1,1)$ by Step 2 and the definition of e^*. So this is a profitable deviation. Thus a non-trivial component and a link-component can not coexist. Steps 1 and 3 together complete the proof.

Characterization of trivial stable networks

Proposition 7. *Under negative link-externalities, trivial stable networks are characterized as follows. A network of multiple link-components and multiple node-components is stable iff* $U(1,1) = U(0,0) \geq U(2,3)$*. A network of multiple link-components and at most one node-component is stable iff* $U(1,1) \geq \max\{U(0,0), U(2,3)\}$*. A network of one link-component* *and multiple node-components is stable iff $U(1,1) = U(0,0) \geq U(2,2)$. A network of multiple node-components is stable iff $U(0,0) \geq U(1,1)$. (Straightforward.)*

Proof of Theorem 2

Lemma 2. *Under positive-link externalities, a connected network is stable iff it is a 1-stable network, and no pair of nodes can strictly improve their payoff by adding a link without cutting any links.*

Proof. "Only if" part is true by definition. For the "if" part, take a network and suppose that a pair i and j has a profitable deviation to G'. Denote S_i the set of links, except the link $\{i,j\}$, that i cuts in the deviation from G to G'. Define S_j analogously.

Suppose that the deviation to G' does not involve adding the link $\{i,j\}$, i.e. either $\{i,j\} \notin E_G, E_{G'}$ or $\{i,j\} \in E_G$. Then G'' where $E_{G''} = E_{G'} \cup S_j$ is a unilateral feasible deviation by i from G. Notice that $d_{i,G'} = d_{i,G''}$ and $e_{i,G'} \geq e_{i,G''}$. Thus $u_{i,G''} \geq u_{i,G'} > u_{i,G}$. This is a contradiction because G is $1-$stable.

Next, suppose that the deviation to G' involves adding the link $\{i,j\}$, i.e. $\{i,j\} \in E_{G'} \backslash E_G$. Since no pair of nodes can strictly improve their payoff by adding a link without cutting any links in G, we can not have $S_i = S_j = \emptyset$. Then w.l.o.g. take $S_i \neq \emptyset$. If $|S_i| = 1$ then $u_{i,G} = u_{i,G'}$ which contradicts G' being a profitable deviation. So $|S_i| \geq 2$.

Since the network G is connected and $\{i,j\} \notin G$, there is a path from i to j in G that does not have the link $\{i,j\}$. Then, there exists a path from some $k \in \{i\} \cup \{i' : \{i,i'\} \in S_i\}$ to j in G that does not use any of the links in S_i.

If $k \neq i$, consider the unilateral deviation by i from G to G'_1 in which i cuts links $S_i \backslash \{i,k\}$. (Note that this is indeed a deviation because $|S_i \backslash \{i,k\}| \geq 1$ implied by $|S_i| \geq 2$.) Clearly $d_{i,G'} = d_{i,G'_1}$. Since there is path from k to j in G that does not include any links in $S_i \backslash \{i,k\}$, k is connected to j in G'_1. Since $\{i,k\} \in E_{G'_1}$, i is connected to j in G'_1. Thus $e_{i,G'_1} \geq e_{i,G'}$. Therefore, $u_{i,G'_1} \geq u_{i,G'} > u_{i,G}$. Contradiction with $1-$stability.

If $k = i$, consider the unilateral deviation by i from G to G'_2 in which i cuts all links in S_i but one. Clearly $d_{i,G'} = d_{i,G'_2}$. Since there is a path from $k = i$ to j that does not involve any links in S_i, i is connected to j in G'_2. Hence $e_{i,G'_2} \geq e_{i,G'}$. Thus $u_{i,G'_2} \geq u_{i,G'} > u_{i,G}$. Contradiction with $1-$stability. □

Lemma 3. *Under positive link-externalities and strict quasi-convexity, a star network with n' nodes is stable iff $U(n'-1, n'-1) \geq U(0,0)$ and $U(1, n'-1) \geq \max\{U(2,n'), U(0,0)\}$*

and a complete network with n' nodes is stable iff $U(n'-1, B(n')) \geq \max\{U(1, 1+B(n'-1)), U(0,0)\}$.

Proof. Call the node in the star that is neighbors with all other nodes the center, and all other nodes leaves. According to Lemma 2, the relevant deviations are type 1– the center cuts some links, type 2– a leaf cuts its link with the center, and type 3– two leaves add a link between. By quasi-convexity, the center does not have a profitable type 1 deviation iff $U(n'-1, n'-1) \geq U(0,0)$. There is no type 2 profitable deviation iff $U(1, n'-1) \geq U(0,0)$. There is no profitable type 3 deviation iff $U(1, n'-1) \geq U(2, n')$.

Suppose that G' is complete. According to Lemma 2, the only relevant deviations are those in which a node cuts some links. By quasi-convexity, such a deviation is not profitable iff $U(n'-1, B(n')) \geq \max\{U(1, 1+B(n'-1)), U(0,0)\}$ □

Lemma 4. *In any network, if there exists a non-central node, there exists at least two non-central nodes. (Straightforward.)*

Lemma 5. *In any network, if there is a central node, all weakly connected nodes must have degree 1. (Straightforward.)*

Lemma 6. *Under positive link-externalities and strict quasi-convexity, a connected stable network is either a star network or a complete network.*

Proof. Take a stable network with n' nodes with e links.

Step 1. An SC node i can cut one link and get $U(d_i - 1, e - 1)$ because it belongs to a cycle. So all SC nodes i must have $U(d_i, e_i) \geq U(d_i - 1, e - 1)$.

Two NC nodes can add a link. Then for at least one of them, the deviation must be non-profitable. This holds for every NC pair. Thus, for all but at most one NC nodes i we must have $U(d_i, e) \geq U(d_i + 1, e + 1)$.

Notice that by the strict quasi-convexity of U, we have $U(d-1, e-1) + U(d+1, e+1) > 2U(d, e)$. Then all but at most one NC nodes must be WC nodes. Otherwise, they want to cut a link unilaterally.

Step 2. Suppose that there exists a C node and an NC node. Then there exists at least two NC nodes by Lemma 4. Since all but one NC nodes are WC (by Step 1), there exists a WC node. We know that there is a C node and a WC node. Recall Lemma 5: whenever there is a C node, all WC nodes must have degree 1. If there were at least two C nodes, then no node could have degree 1. So there is exactly one C node. Then all nodes but one are NC, which means all but at most two are WC by Step 1. Thus, all but at most two nodes have degree 1. In sum, there is one node with degree $n' - 1$, and $n' - 2$ nodes with degree 1. This implies that the remaining node also has degree 1. Therefore, the network is a star network if there exists a C node and an NC node.

Step 3. If there does not exist an NC node, then by definition, the network is complete.

Suppose that there is no C node and so all nodes are NC. Then by Step 1, all but at most one node are WCs. Then there is at most one SC. By Lemma 1, the network is a star network since there exists a unique SC. But then there is a C node since the network is a star. Contradiction.

Combining Steps 3 and 4, we find that the network is either a star or complete network. □

Lemma 7. *If a network is stable, then all of its subnetworks are stable. (Straightforward.)*

Stability part. Now we are ready to complete the proof of the stability part. Take a non-trivial stable network G and denote G' the non-trivial component in G. G' is a connected network by definition. By Lemma 7, G' is stable. By Lemma 6, G' is either a star or complete. Denote n' the order of G'. Suppose that $n' < n$. This means there are some node-components in G along with the non-trivial component G'.

If G' is a star, By Lemma 3, we have $U(n'-1, n'-1) \geq U(0,0)$ and $U(1, n'-1) \geq \max\{U(2,n'), U(0,0)\}$. Then by strict quasi-convexity, we have $U(n', n') > U(n'-1, n'-1) \geq U(0,0)$. So the center of the star strictly improves by forming a link with a node-component. By positive link-externalities, $U(1, n') > U(1, n'-1) \geq U(0,0)$. So a node-component strictly improves by forming a link with the center of the star. This is a contradiction with the stability of G. Thus $n' = n$ and so G is a star. By 3, this is stable iff $U(n-1, n-1) \geq U(0,0)$ and $U(1, n-1) \geq \max\{U(2,n), U(0,0)\}$

If G' is complete, By Lemma 3, we have $U(n'-1, B(n')) \geq \max\{U(1, 1+B(n'-1)), U(0,0)\}$. By strict quasi-convexity, we have $U(n', B(n')+1) > U(n'-1, B(n')) \geq U(1, 1+B(n'-1))$. So any node in the the non-trivial component strictly improves by forming a link with a node in a node-component. Also by positive link-externalities, $U(1, B(n')+1) \geq U(n', (B(n') + 1) > U(n'-1, B(n')) \geq U(0,0)$. So any node in a node-component strictly improves by forming a link with a node in the non-trivial component. This is a contradiction with the stability of G. Thus $n' = n$ and so G is a complete network. By Lemma 3, this is stable iff $U(n-1, (B(n)) \geq \max\{U(1, 1+B(n-1)), U(0,0)\}$. This completes the proof of the stability part.

Strong stability part. Consider a stable star network G. Denote i_0 the center node.

Step 1. First consider feasible deviations that involve i_0. Take such a feasible deviation say G_1. If G_1 is a unilateral deviation by i_0, it can not be profitable, by stability. Let G_1 involve some leaves, say $D \subset N\backslash\{i_0\}$. Consider the deviation G_2 by D such that $E_{G_2} \cap [D \cup \{i_0\}]^2 = E_{G_1} \cap [D \cup \{i_0\}]^2$ and $E_G\backslash[D \cup \{i_0\}]^2 \subset E_{G_2}$. This is a feasible deviation by D because 1) nodes in D are neighbors with i_0 in G, and so G_1 can not involve adding a non-existing link between i_0 and D, and 2) if some links between i_0 and D were cut in G_1, these links can still be cut by D. Clearly $d_{i,G_1} = d_{i,G_2}$ and $e_{i,G_2} \geq e_{i,G_1}$ for all $i \in D$. Therefore, if G_1 is a profitable deviation, then G_2 is also a profitable deviation. That is, if there exists a profitable deviation that involves the center, then there exists a profitable deviation that does not involve the center. So there exists a profitable deviation iff there exists a profitable deviation by a subset of the leaves. Call the latter *leaf-deviations*.

Step 2. A deviation by some leaves that involves at least one leaf cutting the link with i_0 and not adding any links can not be profitable, by stability. So in any profitable deviation by some leafs, all deviators who cut their link with i_0 must have added some links with other deviators.

Consider the/a leaf-deviation in which the minimum number of links between the deviators and the center are cut, if leaf-deviations exists. Denote this deviation G_1. If there exists a deviator, say i_1, that indeed cut its link with i_0, it must have added at least one link with another deviator, say i_2, by Step 1. Then consider the feasible deviation G_2 by N_{G_1} given by $E_{G_2}\backslash E_{G_1} = \{i_0, i_1\}$ and $E_{G_1}\backslash E_{G_2} = \{i_1, i_2\}$. It is clear that G_2 is a profitable deviation since G_1 is. But G_2 involves cutting less links with i_0. This is a contradiction with the definition of G_1. Thus, there are no deviators in G_1 that cut the link with i_0.

All in all, there exists a leaf-deviation iff there exists a leaf-deviation in which no deviators cut their link with i_0. Call the latter *connected-leaf-deviations*.

Step 3. Consider the/a connected-leaf-deviation in which the deviators have the largest number of edges in their component (which is the whole network) after the deviation, if any connected-leaf-deviation exists. Call this deviation G_1. Denote e the number of links in the network. Take one deviator, say i, that has the largest degree among all deviators after the deviation. It holds that $U(1, n-1) < U(d_{i,G_1}, e)$ since this is a profitable deviation. If there exists two deviators j, j' such that $d_{j,G_1}, d_{j',G_1} < d_i$, then consider the deviation G_2 by N_{G_1} given by $E_{G_2} = E_{G_1} \cup \{j, j'\}$. The number of links in the component of deviators is now $e+1$. So for $j'' \in N_1\backslash\{j, j'\}$, $u_{j'',G_2} > u_{j'',G_1} > u_{j'',G}$. As for j and j', $u_{j/j',G_2} = U(d_{j/j',G_2}, e+1) = U(d_{j/j',G_1}+1, e+1) \geq U(d_{i,G_1}, e+1) > U(d_{i,G_1}, e) > u_{i,G} = u_{j/j',G}$. Thus, G_2 is a connected-leaf-deviation, but G_2 has more links than G_1, which is a contradiction. Therefore all but at most one deviator has the same degree after the deviation G_1.

Let all but one deviators in G_1 have degree d, and one other, say j''_0 has degree $d' \leq d$. Suppose that $d' \leq d-2$. We first claim that two deviators $j''_1, j''_2 \notin N_{j'',G_1}$ can not be neighbors of each other in G_1. Otherwise the deviation G_2 by N_{G_1} given by $E_{G_2} \backslash E_{G_1} = \{\{j'', j''_1\}, \{j'', j''_2\}\}$ and $E_{G_1} \backslash E_{G_2} = \{\{j''_1, j''_2\}\}$ would be connected-leaf-deviation that involves more links than G_1. Therefore, deviators in $N_{G_1} \backslash N_{j''_0, G_1}$ are not neighbors of each other. But then, they can not have larger degree than j'', which is a contradiction. So we must have $d' \geq d - 1$. (Note that $d' = d - 1$ iff $(d-1)|N_{G_1}|$ is odd.) Then notice that $e_{G_1} = n - 1 + \lfloor (0.5)(d-1)|N_{G_1}| \rfloor$.

Given these, suppose that $N_{G_1} \neq N$. Then the payoffs of all but one deviators are $U(d, e_{G_1}) > U(1, n-1)$. Take some $i' \notin N_{G_1}$. Consider a deviation G_3 by $N_{G_1} \cup \{i'\}$ wherein one no deviator cuts any links, one deviator has degree d or $d+1$, and all other deviators have degree $d+1$.[3] Then the deviators have payoff at least $U(d, n-1+\lfloor (0.5)(d-1)(|N_{G_1}|+1) \rfloor) \geq U(d, e_{G_1}) > U(1, n-1)$. Thus G_3 is a connected-leaf-deviation which involves more links than G_1. This is a contradiction so we must have $N_{G_1} = N$.

So, there exists a connected-leaf-deviation iff there exists a d and connected-leaf-deviation by all leaves in which all but one node deviators have degree d and the other has degree d or $d-1$ (depending on the parity of $(d-1)(n-1)$) after the deviation. Call the latter *$d-$regular-connected-leaf-deviations.*

Step 4. After a $d-$regular-connected-leaf-deviation, the payoff of deviators is $U(d, n-1+\lfloor (0.5)(d-1)(n-1) \rfloor)$. The potential residual deviator has higher payoff so his incentive constraint does not bind. Therefore, combining all steps, we find that G is strongly stable iff $U(1, n-1) \geq U(d, n-1+\lfloor (0.5)(d-1)(n-1) \rfloor)$ for all d. By strict quasi-convexity, this holds iff $U(1, n-1) \geq U(n-1, B(n))$.

Next consider a stable complete network. By $1-$stability, $U(d, B(n)-(n-1-d)) \leq U(n-1, B(n))$ for all d. In any feasible deviation by any subset of nodes after which a deviators degree becomes d, the number of links in the component is at most $B(n)-(n-1-d)$. So there are no profitable deviations.

Proof of Proposition 3

The part about stability follows after checking relevant deviations and some algebra. For efficiency, consider any network. Index components by t. Let e_t and k_t be the number of links and nodes in component t. The sum of payoffs is $\sum_t e_t(2 - ck_t)$. Thus, it is inefficient to have a component with $k_t > \frac{2}{c}$. For $k_t < \frac{2}{c}$, it is efficient to have all links possible in the component. Thus the efficient network must consist of disjoint cliques. Then the objective is

[3] For any k and $1 \leq d'' \leq k-1$, if kd'' is even there exists d''regular network on k nodes and if kd'' is odd, there exists a network where one node has degree $d''-1$ and all the others have degree d''. This is a consequence of Erdos-Gallai Theorem.

to maximize the average payoff in a network that consists of disjoint cliques, which coincides, by definition, with strongly stable networks.

Proof of Proposition 4

Take a network. Index components by t. Let e_t and k_t be the number of links and nodes in component t. By additivity, the sum of payoffs in component t is $U(2e_t, e_t)$ which depends only on the number of links. Thus, $e_t = B(k_t)$ is efficient. Therefore, efficiency requires maximizing $\sum_t k_t U(k_t - 1, B(k_t))$ subject to $\sum k_t = n$. Then the solution is $k_t = d^* + 1$ for all t as $d^* = \arg\max U(d, B(d+1))$.

If $U(2x, x)$ is not an increasing function of x, it is still efficient to make components "dense" in the sense that if $e_t < B(k_t)$, then less than k_t nodes can be used to form a component with e_t links. The residual nodes can be used elsewhere or alone to increase total payoff as $U(0, 0) > 0$. Thus, e_t must be between $B(k_t - 1)$ and $B(k_t)$. That is, the components must be "close" to a clique with perhaps at most $k_t - 2$ missing links. Therefore, efficiency roughly requires maximizing $\sum_t k_t U(k_t - 1, B(k_t))$

Proof of Proposition 5

For $c < 1$, every pair would add their missing link. So the unique stable network is the complete network. By Theorem 2, this is strongly stable. For $c > 1$, every pair would cut their existing link. So the unique stable network is the empty network. For $c < \frac{n}{2}$, this is not strongly stable because all agents would jointly deviate to the complete network. So there are no strongly stable networks. For $c > \frac{n}{2}$, consider a deviation from the empty network and take a component with k nodes and e links. The sum of payoffs is $ke - 2ce < 0$ meaning that at least one deviator has negative payoff. So this can not be a profitable deviation. So the empty network is strongly stable. Finally, consider efficiency. The sum of payoff in a component is $e(k - 2c)$. If $n < 2c$, this can not be made positive, so the unique stable network is the empty network. If $n > 2c$, there can be components with more than $2c$ nodes in the efficient network. But combining them into the complete network clearly increases the total payoff.

Proof of Proposition 6

First we show that an efficient network must be connected. Suppose not. Take two components. If either are node-components, take two neighbors i_1, i_2 from one component and two neighbors j_1, j_2 from the other. Cut links $\{i_1, i_2\}$ and $\{j_1, j_2\}$, add links $\{i_1, j_1\}$ and $\{i_2, j_2\}$. Then, the degrees of all nodes are unchanged. The sum of payoffs clearly increase as the components are combined. If one is a node-component and the other G' is not, take i from the node-component and j from the other, and add the link $\{i, j\}$. The change in the sum of payoffs is

$$\frac{1}{2}(\sum_{j' \in N'} d_{j'} + 2)(\sum_{j' \in N'} \frac{1}{d_{j'}} + 1 - \frac{1}{d_{j'}(d_{j'} + 1)}) - \frac{1}{2}(\sum_{j' \in N'} d_{j'})(\sum_{j' \in N'} \frac{1}{d_{j'}}) > 0$$

So the network must be connected. Now we are interested in finding a degree sequence $\{d_i\}_{i=1}^n$ for a connected graph to maximize:

$$(\sum_{i=1}^n d_i) \sum_{i=1}^n \frac{1}{d_i}.$$

For the moment, let us fix the number of edges in the graph to be m and consider the following problem:

$$\max m \sum_{i=1}^n \frac{1}{d_i}$$

subject to

$$\sum_i d_i = m$$

$$d_1 \geq d_2 \geq \ldots \geq d_n$$

$$\sum_{i=1}^r d_i \leq r(r-1) + \sum_{i=r+1}^n \min(d_i, r) \; \forall r$$

$$n - 1 \geq d_i \geq 1 \; \forall i$$

This is the problem of finding among all m-edge graphs with n nodes, the one with the smallest $\sum_{i=1}^n \frac{1}{d_i}$. This problem has been investigated in Hu et al. (2007). Their Theorem 3.5 tells us that we can describe the optimal graph for the problem just stated using just two parameters k and $a \in [0, k]$. The graph will consist of a single clique of size k, one of the vertices in the clique will have degree $n - 1$; call it vertex 1. There will be $n - k - 1$ one degree vertices all connected to vertex 1 and a single vertex with degree a whose neighbors are in the clique and include vertex 1. We can describe the degree sequence of the graph in the following way.

1. $d_1 = n - 1$

2. $d_j = k$ for $2 \leq j \leq a$

3. $d_j = k-1$ for $a+1 \leq j \leq k$

4. $d_{k+1} = a$

5. $d_j = 1$ for $k+2 \leq j \leq n$

This means that $m = k(k-1)/2 + a + n - k + 1$. Observe that if $a = 0, 1, k$, this graph is a pineapple.

Now lets make m a variable. The goal will be to show that for an optimal choice of m we would set $a = 0, 1, k$. Our objective function is

$$f(a) = [k(k-1) + 2a + 2(n-k) + 2][\frac{1}{n-1} + \frac{a-1}{k} + \frac{k-a}{k-1} + \frac{1}{a} + n - k - 1]$$

Let $A = k(k-1) + 2(n-k) + 2$ and $B = \frac{1}{n-1} - \frac{1}{k} + \frac{k}{k-1} + n - k - 1 = \frac{1}{n} + \frac{1}{k(k-1)} + n - k$. Then, our objective function is

$$(A + 2a)(B - \frac{a}{k(k-1)} + \frac{1}{a}).$$

Now

$$B - \frac{a}{k(k-1)} + \frac{1}{a} \geq B - \frac{1}{k-1} + \frac{1}{k} = \frac{1}{n-1} - \frac{1}{k} + \frac{k}{k-1} + n - k - 1 - \frac{1}{k-1} + \frac{1}{k} = \frac{1}{n-1} + n - k$$

Hence $f(a) \geq (A + 2a)(\frac{1}{n-1} + n - k)$ which is increasing in a. We show that f is convex.

$$f'(a) = 2(B - \frac{a}{k(k-1)} + \frac{1}{a}) - (A + 2a)(\frac{1}{k(k-1)} + \frac{1}{a^2})$$

$$\begin{aligned} f''(a) &= -\frac{2}{k(k-1)} - \frac{1}{a^2} - \frac{2}{k(k-1)} + \frac{2A}{a^3} + \frac{2}{a^2} \\ &= -\frac{4}{k(k-1)} + \frac{2A}{a^3} + \frac{1}{a^2} \end{aligned}$$

For $a \leq k-1$ we have that

$$f''(a) \geq -\frac{4}{k(k-1)} + \frac{2A}{k(k-1)^2} + \frac{1}{k(k-1)}$$

$$= -\frac{3}{k(k-1)} + 2\frac{k(k-1) + 2(n-k) + 2}{k(k-1)^2}$$

$$= -\frac{3}{k(k-1)} + \frac{2}{k-1} + 4\frac{(n-k) + 1}{k(k-1)^2} \geq 0$$

So, the function f is convex. As it lies above a linear function that is increasing, it follows that $f(a)$ cannot have a maximum at some value of a strictly between 1 and k.

Therefore, the efficient network must be a pineapple graph, say with m nodes in the clique. The remaining $n - m$ nodes are neighbors only with the center node. The value of this network is

$$\left(\binom{m}{2} + n - m\right)\left(\frac{n}{n-1} + n - m\right)$$

An analytical solution exists as this is a cubic in m but the expression is extremely long and hard to interpret. For large n let $m = n\alpha$ where $\alpha \in [0, 1]$. In the limit with respect to n,

$$\begin{aligned}
&\left(\binom{m}{2} + n - m\right)\left(\frac{n}{n-1} + n - m\right) \\
\approx &\left(\frac{n^2}{2}\alpha^2 + n(1-\alpha)\right)\left(\frac{n}{n-1} + n - n\alpha\right) \\
= &n^3\left(\frac{1}{2}\alpha^2 + \frac{1-\alpha}{n}\right)\left(\frac{1}{n-1} + 1 - \alpha\right) \\
\approx &n^3\left(\frac{1}{2}\alpha^2\right)(1-\alpha)
\end{aligned}$$

which is maximized for $\alpha = \frac{2}{3}$.