# Network Formation and Systemic Risk*

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**Abstract**

This paper introduces a simple model of endogenous network formation and systemic risk. In the model, firms form joint ventures called 'links' which are subsequently subjected to shocks that are either good or bad. Bad shocks incentivize default. Links yield full benefits only if the counter-party does not subsequently default on the project. Accordingly, defaults triggered by bad shocks render firms insolvent and defaults propagate via links. The model yields three insights. First, stable networks with ex-ante identical agents exhibit a core-periphery structure. Second, an increase in the probability of good shocks increases systemic risk. Third, the network formed critically depends on the correlation between shocks to links. As a consequence, an observer who misconceives the correlation will significantly underestimate the probability of systemwide default.

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*Keywords*: Network Formation, Systemic Risk, Core-periphery, Volatility Paradox, Group Stability.

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# 1 Introduction

The awkward chain of events that so upset the bankers in 2008 began with the collapse of Lehman Brothers. Panic spread, the dollar wavered, and world markets fell. Interconnectedness of the financial system, it was suggested, allowed the Lehman Brothers' fall to threaten the stability of the entire system. The possibility that the failure of one firm will trigger the widespread failure of otherwise healthy firms is called systemic risk. Even the failure of a non-financial firm, such as General Motors, will, via its suppliers, have spillovers into the non-automotive sector. Indeed, the 2009–2014 restructuring plan filed by General Motors says:

> "The systemic risk to the automotive industry and the overall U.S. economy are considerable, just as the bankruptcy of Lehman had a ripple effect throughout the financial industry ..."

These events inspired scholars to characterize network structures that are conducive to systemic risk, which is the risk that shocks to a part of the system propagate and damage the entire system. With some exceptions, these papers assume an exogenously given network. A node (or subset of them) is subjected to a shock and the propagation of the shock across the network is studied. Absent are reasons for the presence of links between nodes. In this paper, a link between two nodes represents a potentially lucrative joint project. However, each link increases the possibility of contagion. In the presence of a trade-off between being exposed to systemic risk and having more projects, we ask what kinds of networks would be formed? Systemic risk is not unique to financial networks. It is a concern in supply chains and the web of firms linked by joint projects or trade credits. In these examples, other externalities are present, but they are *not* a focus of this paper.

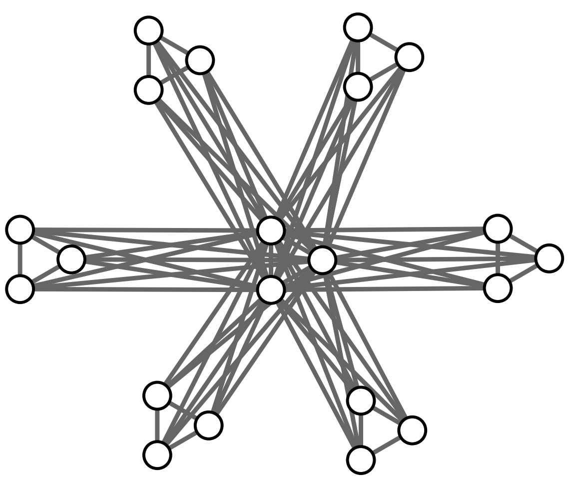
We propose a *simple* model of contagion that spreads across agents whom we call firms. In the model, firms form links and become counter-parties. Links represent joint ventures, and each link is subjected to a good or a bad shock. We assume the networks formed are stable, in the sense that no subset of firms has an incentive to deviate and choose a different set of links. A formal definition appears in Section 2.2.

A bad shock to a link makes firms incident to that link default. Moreover, any firm that has a defaulting counter-party also defaults. We also allow direct shocks to firms (called node shocks), which can be interpreted as idiosyncratic risks of default. Systemic risk is the probability of the event that every firm chooses to default. Subsection 2.1.2 contains a detailed micro foundation of this model in terms of firms borrowing from outside lenders and investing as pairs into projects. The main insights are enumerated below.

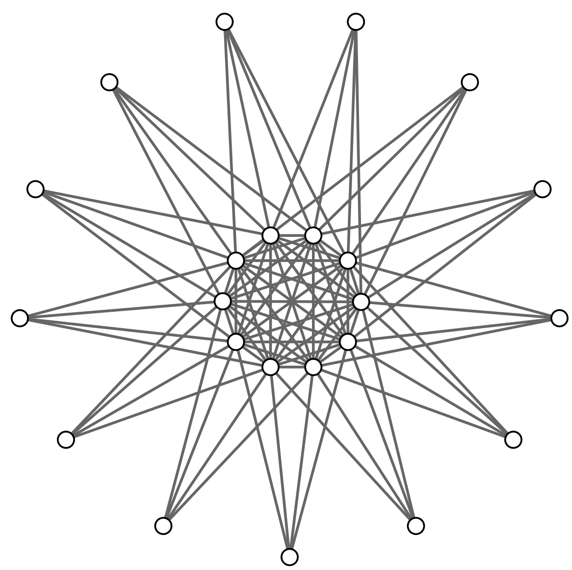
**1. Core-periphery networks emerge despite systemic risk.**

Core-periphery networks are considered ubiquitous; see, for example, Bech and Atalay (2010) and Craig and Von Peter (2014). They consist of a subset of nodes (the core) that are densely connected to each other and a periphery of nodes partitioned into clusters that are all connected to the core but 'lightly' connected to each other. Examples of such networks are displayed in Figure 1. Prior work has shown how core-periphery networks can emerge endogenously but not when systemic risk is a concern (except Farboodi (2015) and Erol (2017)). Such structures may facilitate systemic risk because the core, with its dense interconnections, would encourage contagion. Financial institutions that occupy the core, for example, are thought to be contributors to systemic risk for this very reason.[1]

In this paper, stable networks exhibit a core-periphery structure *even* when firms are ex-ante identical. They arise because of a coordination failure, as described in section 4. In the general case, firms better able to withstand counter-party failures and link shocks make up the core. In this sense, the core acts as a barrier to the spread of contagion between distinct components in the periphery. However, each firm, including those in the core, is susceptible to failure caused by idiosyncratic node shocks. Accordingly, the core makes the default risk of the entire system highly correlated and becomes a major source of systemic risk.



Small core



Large core

Figure 1: Structure of the stable network: core-periphery

**2. Peripheral nodes are organized into disjoint cliques that are stable and effi-**

[1]High level of interconnectedness of certain institutions is seen as systemic threat. See, for example, http://www.fsb.org/what-we-do/policy-development/systematically-important-financial-institutions-sifis/

**cient.**

A clique is a group of firms with links to each other. In our model, the peripheral firms of a stable network are partitioned into cliques, as illustrated in Figure 1. This limits the risk of contagion among peripheral firms. The disjoint clique structure in the periphery is efficient in the sense of maximizing total expected payoff of the firms.

In our model, stability and efficiency coincide, in contrast to Jackson and Wolinsky (1996), which notes a cleavage between pairwise stability and efficiency. This is because in our model, 'distant' links confer no benefit and serve only as a channel for contagion.

**3. Volatility Paradox**

If the probability of a good shock to a link is increased, the network becomes more interconnected at a rate at which systemic risk always increases. This is a network version of the volatility paradox of Brunnermeier and Sannikov (2014): low volatility leads investors to behave in ways that make the financial system more fragile and prone to crisis. We show that the volatility paradox persists, even when contagion is not as strong and network externalities are mild. In the absence of contagion there is no volatility paradox. Therefore, the volatility paradox is present in our setting only because of contagion.

**4. Systemic risk vs. systematic risk: What can we learn from the network about the probability of system-wide failure?**

Systematic risk is the risk of system-wide failure due to common exposures of institutions to risks outside the system, whereas systemic risk is the risk that the system fails via the propagation of bad shocks. When shocks are perfectly correlated, i.e. under systematic risk, the network formed is strongly interconnected. When shocks are idiosyncratic, i.e. under systemic risk, such a strongly interconnected network is formed only when shocks are likely to be good. Accordingly, an endogenously formed and strongly interconnected network implies an upper bound on the underlying systemic risk, whereas such a network is uninformative about the underlying systematic risk. An outsider who observes a very dense network must first understand the correlation structure before assessing the probability of system-wide failure.

We think this relevant to the debate between two theories of financial destruction advanced to explain the 2008 financial crisis. The first, described above, is dubbed the 'domino theory' of systemic risk. The alternative, advocated by Edward Lazear,[2] is dubbed the 'popcorn' theory of systematic risk. Lazear describes it thusly in a 2011 opinion piece in the *Wall Street Journal*:

[2]Chair of the US President's Council of Economic Advisers during the 2007-2008 financial crisis.

> “The popcorn theory emphasizes a different mechanism. When popcorn is made (the old-fashioned way), oil and corn kernels are placed in the bottom of a pan, heat is applied and the kernels pop. Were the first kernel to pop removed from the pan, there would be no noticeable difference. The other kernels would pop anyway because of the heat. The fundamental structural cause is the heat, not the fact that one kernel popped, triggering others to follow.
>
> Many who believe that bailouts will solve Europe’s problems cite the Sept. 15, 2008 bankruptcy of Lehman Brothers as evidence of what allowing one domino to fall can do to an economy. This is a misreading of the historical record. Our financial crisis was mostly a popcorn phenomenon. At the risk of sounding defensive (I was in the government at the time), I believe that Lehman’s downfall was more a result of the factors that weakened our economic structure than the cause of the crisis.”

### Related literature

This paper contributes to four streams in the economic analysis of networks. We summarize them here. Detailed comparisons to prior work can be found in the body of the paper.

**Systemic Risk in Networks:** Much prior work, such as Acemoglu et al. (2015), Eboli (2013), Elliott et al. (2014), Gai et al. (2011), and Glasserman and Young (2015), takes the network as exogenous. We consider fully strategic network formation. Acemoglu et al. (2015) contains a discussion of network formation, but within a set of limited alternatives. Babus (2016) also has a model of network formation, but one in which agents share the goal of minimizing the probability of system-wide default. In our model, agents are concerned with their own expected payoffs and only indirectly with the possibility of system wide failure. Our network formation model is closest to Blume et al. (2013). However, in that paper, the risk of a node initially defaulting to start contagion is independent of the network formed because the exogenous shocks hit nodes. In our model, the likelihood of a node initially defaulting to start contagion depends on the network, in particular, the degree of the node, because the exogenous shocks hit edges.

**Contagion in Networks:** Our paper is also a contribution to the literature on contagion in networks (see, for example, Morris (2000) and Goyal and Vega-Redondo (2005)). It extends this literature by endogenizing the networks and incorporating uncertain payoffs.

**Core-Periphery Structure:** Many financial networks exhibit a core-periphery structure. A variety of explanations have been offered. Farboodi et al. (2017), for example, shows that a core-periphery structure emerges in a search model where some agents choose to trade faster than others. Wang (2016) demonstrates that a core-periphery structure emerges in an inter-dealer market with ex-ante identical agents as the result of a tradeoff between trade competition and inventory efficiency. None of these papers account for contagion and systemic risk. Two papers that do are Farboodi (2015) and Erol (2017). The agents in these papers are ex-ante heterogenous. This paper generates a core-periphery structure with ex-ante identical agents.

**Network Formation:** This literature examines the networks that emerge in equilibrium when agents' payoffs are a function of the network. These payoff functions can be interpreted as the reduced form of a more complicated interaction that is not formally modeled. Our network formation results hold for any setting where an agent's payoffs can depend arbitrarily on her own degree but are strictly decreasing in the number of links in the connected component she is a member of. In this paper such a property arises as a consequence of contagion. However, other concerns can give rise to them. For example, a seller might prefer to deal with buyers who are linked with few other sellers and vice-versa.

**Structure**

For ease of exposition, we introduce the model from the 'outside in'. In Section 2, we focus on ex-ante identical firms and link shocks only. This generates the structure that characterizes the periphery in the general model. We study the stability and efficiency of the periphery, and peripheral systemic risk first. Subsection 2.1.2 shows how to interpret the links and the associated payoffs as arising from joint ventures. It can be omitted without loss of continuity.

In Section 3, we introduce node shocks and firms with varying degrees of resilience to shocks. The firms that are more resilient to contagion will end up in the core of the network. We study the stability and efficiency of the core-periphery structure, and central systemic risk. Subsection 3.1.2 extends the foundation given in 2.1.2 to node shocks.

Section 4 builds on subsections 2.1.2 and 3.1.2 to show how a core-periphery structure can emerge even with ex-ante identical agents. Section 5 concludes.

# 2 The Periphery

## 2.1 The Model of the Periphery

### 2.1.1 Model of contagion

Denote by $N = \{1, 2, ..., n\}$ a finite set of *firms*. In what follows we will sometimes refer to firms as nodes. A *link* $\{i, j\}$ represents a joint venture between firms $i$ and $j$. $E \subset [N]^2$ is the set of links and $(N, E)$ is the *network*. If $\{i, j\} \in E$, then $i$ and $j$ are called *counter-parties*. Denote by $D_i = \{j \in N | \{i, j\} \in E\}$ the set of counter-parties of $i$ and $d_i = |D_i|$ the *degree* of $i$. There exists $\tilde{d} \leq n$ such that $d_i \leq \tilde{d}$ for all $i$. For each link $\{i, j\} \in E$, two outcomes $s_{ij}$ and $s_{ji}$ are chosen by nature identically and independently. $s_{ij}$ represents the net benefit of firm $i$ in its partnership with $j$. With probability $\alpha \in (0, 1)$, the link is *good for* $i$: $s_{ij} = \theta$ where $\theta > 0$ is a constant. With probability $1 - \alpha$, the link is *bad for* $i$: $s_{ij} = \theta - \gamma$ where $\gamma > 0$ is a constant. Denote by $b_i$ the number of links that are bad for $i$.

Upon the realization of shocks, each firm $i$ either *continues* or *defaults*. If $i$ continues, it receives the sum of $\sum_{j \in D_i} s_{ij}$. Furthermore, for each $j \in D_i$, if $j$ defaults, $i$'s benefit from its link with $j$ is reduced by $\beta > 0$, from $s_{ij}$ to $s_{ij} - \beta$. This captures the extra costs that must be incurred to cover for $j$'s role in the project. Denote the number of defaulting counter-parties of $i$ by $f_i$ . $i$'s payoff from continuation is $u_i = d_i\theta - f_i\beta - b_i\gamma$. If $u_i$ is non-negative, $i$ continues. If $u_i$ is negative, $i$ defaults and gets an outside option 0. Losses due to bad shocks ($\gamma$'s) can trigger defaults, which can propagate via counter-party losses ($\beta$'s). This describes the contagion, which is exogenous for now.

As we wish to focus on systemic risk only, we make an assumption that ensures that contagion is the only major risk for firms.

**Assumption 1.** *Contagion starts and spreads very easily:* $\tilde{d}\theta < min\{\beta, \gamma\}$.

Recall that $d_i \leq \tilde{d}$ for all $i \in N$. $\tilde{d}\theta < \beta$ implies that firms are heavily exposed to each other and contagion spreads easily. One defaulting counter-party makes the firm drop all projects. $\tilde{d}\theta < \gamma$ implies that firms are individually fragile and contagion starts easily. One bad shock makes the firm drop all projects. All firms that are connected[3] face a risk of contagion, and this is the only risk they face. The links provide no diversification benefit.[4] The consequences of relaxing Assumption 1 are discussed in Appendix C.

**Proposition 1.** *The expected payoff of a firm with degree $d$ and whose component has $e$ links is given by*

$$U(d, e) := d\alpha^{2e}\theta. \tag{1}$$

[3]We recall some standard definitions from graph theory. $i$ and $j$ are called *adjacent* if they have a link. Adjacent firms are counter-parties in our model. If $i$ and $j$ are adjacent, we say $i$ and $\{i, j\}$ are *incident*. $i$ and $j$ are called *connected* if one can be reached from the other via a sequence of links. The *component* of $i$ is the set of all firms that are connected to $i$ and all links between these firms. A *clique* is a collection of firms such that all pairs have links with each other. A *disjoint clique* a component which is a clique. The *order* of a clique is the number of firms in the clique. A $(d+1)-$*clique* is disjoint clique of order $d+1$. Note that each firm in a $(d+1)$-clique has degree $d$.

[4]Elliott et al. (2014) call this a network of full integration. They study the tradeoff between integration and diversification but do not characterize the network formed. Although the models are different, our paper can be seen as the network formed in the corner case of full integration and no diversification.

A firm with $d$ counter-parties and belonging to a component that has $e$ links has expected payoff $U(d, e)$. Many of our main results use *only* the fact that $U(d, e)$ is strictly decreasing in $e$, not the particular functional form in expression (1). Accordingly, our results can be applied to more general network settings, wherein the payoff of a node is any function of $d$ and $e$ that is decreasing in $e$.

### 2.1.2 Meaning of the model

We now make our interpretation of the links formed as joint ventures concrete. This section as well as the companion section 4 can be omitted without loss of comprehension. Assumptions and propositions related to this section are flagged with the letter 'M'.

Each link is a joint project funded by loans obtained from outside lenders. The project requires a monetary investment from each counter-party and costly effort to be exerted before maturity of the project. Firms are free to drop any projects in their portfolio by not exerting effort. Dropped projects yield no return to the counter-party that dropped the project. Moreover, the return from a project is reduced if the counter-party does not exert effort and drops the project. In our model, the cost of effort for the project $\{i, j\}$ is both uncertain and independent across the counterparts. This can be justified in at least two ways.

- The partners are responsible for different and independent parts of the project and the costs they bear are uncertain. This is not at all unusual in, for example, construction.
- The project may have (positive or negative) externalities on projects/products that do not belong to the scope of the joint venture (see Dessein (2005), Kumar (2010)).

While we assume that $s^{ij}$ and $s^{ji}$ are independent of each other, the results easily carry over to the case when they are perfectly correlated or even conditionally independent. This can be interpreted as a common cost or payoff shock to the counter-parties.

Formally, firms first borrow from outsiders and use these borrowed funds to invest jointly in pairs. Each firm has access to a credit line from its outside lender of capacity $L$. Each firm is specialized, so bilateral partnerships are necessary to invest. Each project costs $P$ to each counter-party as an initial investment. Accordingly, a firm that has $d$ counter-parties borrows $dP$ in total out of its credit line $L$ and promises to pay $R$ per unit borrowed. (We endogenize $R$ later in section 4. All results go through with endogenous $R$ as well.) The number of links a firm can form is bounded by $\tilde{d} = L/P$. The investment choices correspond to network formation. The solution concept for network formation is described in Section 2.2.

After investment, each firm must bear a management cost for the project. The effectiveness of each firm in each project is random. Firm $i$'s cost for the project with firm $j$ needs $C^{ij} \in \left\{\overline{C}, \underline{C}\right\}$ cost of effort from $i$. $C^{ij} = \underline{C}$ with probability $\alpha \in (0, 1)$ and $C^{ij} = \overline{C}$ with probability $1 - \alpha$, where $\underline{C} < \overline{C}$. (These random costs correspond to shocks $s_{ij}$'s.) Note that managements costs are in terms of utilities, such as effort costs.

Upon realization of the shocks, each firm simultaneously decides which among the projects it has invested in to *keep* and which to *drop*. If $i$ decides to drop the project with $j$, $i$ receives nothing from the project, but also avoids the management cost $C^{ij}$. If $i$ continues with the project with $j$, $i$ incurs the management cost $C^{ij}$. $i$'s return depends on the effort of $j$. If $j$ also continues with the project, $i$ gets $\overline{B}$ revenue from the project. If $j$ drops the project, $i$ gets $\underline{B}$ revenue from the project, where $\underline{B} < \overline{B}$. Denote

$$\theta = \overline{B} - \underline{C} - PR,\ \beta = \overline{B} - \underline{B},\ \gamma = \overline{C} - \underline{C}. \tag{2}$$

**Assumption M 1.** *Projects are ex-post profitable:* $\underline{B} > \overline{C}$.

**Proposition M 1.** *The best response of a firm is either to keep all projects or to drop all projects. If $i$ drops all projects, its payoff is* $0$. *If $i$ keeps all projects, its payoff is*

$$u_i = d_i\theta - f_i\beta - b_i\gamma.$$

We allow a firm to drop some projects and keep others, but it is never optimal to do so. Since projects are ex-post profitable, dropping or keeping projects is a matter of the ability of the firm to repay its debt $PR$ and still be compensated for the management costs. If the total proceeds from all projects covers the management costs and the loan, then the firm keeps all projects because each project adds ex-post value to the firm. Otherwise, it is impossible for the firm to obtain positive value, and so the firm drops all projects, saves management costs, and earns 0.

Fix a network and a realization of shocks. Consider the game where the firms simultaneously decide which of their various projects they keep or drop. There are potentially many Nash equilibria of this game. As the game is supermodular, there exists a "best Nash equilibrium'," in the sense that any firm who keeps some projects in at least one Nash equilibrium keeps these projects in the "best Nash equilibrium" as well. Call this the *cooperating equilibrium.*

**Assumption M 2.** *Interest rates are high, and so profit margins are small:*

$$0 < L\left(\frac{\overline{B} - \underline{C}}{P} - R\right) < min\left\{\left(\overline{B} - \underline{B}\right), \left(\overline{C} - \underline{C}\right)\right\}.$$

Let $\tilde{d} = L/P$. Then, $\tilde{d}$ is the maximum number of links a firm may have because firms cannot borrow more than $L$ and each project costs $P$ to initiate.[5]

## 2.2 Stability and Efficiency of the Periphery

### 2.2.1 Stability

As network formation occurs prior to the realization of shocks, firms evaluate each network $(N, E)$ with respect to their expected payoffs in the continuation. The payoffs in the continuation are given by the expression (1). There is an upper bound $\tilde{d} \leq n - 1$ such that firms cannot form more than $\tilde{d}$ links. Consider a candidate network $(N, E)$ and a subset of firms $N' \subset N$. A *feasible deviation* by $N'$ allows firms in $N'$ to *simultaneously*

1. add absent links within $N'$,

2. delete any link incident to at least one vertex in $N'$.

A *profitable deviation* by $N' \subset N$ is a feasible deviation that strictly improves the expected payoff of each member of $N'$. $(N, E)$ is called a *group stable* if there is no profitable deviation for any $N' \subset N$.[6] $(N, E)$ is *bilaterally stable* if there are no profitable deviations for any $N' \subset N$ with $|N'| \leq 2$. We first characterize bilaterally stable networks, then group stable networks, and then examine their efficiency.

### 2.2.2 Bilaterally stable networks

**Proposition 2.** *Any bilaterally stable network consists of disjoint cliques.*

Proposition 2 is unsurprising. Other papers (such as Blume et al. (2011), Elliott and Hazell (2016), Erol (2017)) have obtained a similar result. Less clear is why the components are

[5] Assumption M 2 is the counterpart of Assumption 1.

[6] In Dutta and Mutuswami (1997) and Erol (2017) this solution concept is called strongly stable. Farboodi (2015) calls this solution concept group stable. We think the second more evocative. Jackson and Van den Nouweland (2005) considers a stronger notion that rules out deviations by any coalition $V$ that weakly improves the payoff of all members, and strictly improves the payoff of at least one member.

cliques. This is because any two firms connected to each other are already exposed to each other's risk of default via contagion. If they have not formed a link yet, by forming this link and deleting one link each with other counter-parties, their respective degrees are unchanged, but they strictly reduce the risk of default. This is a consequence of having link shocks in the model. Therefore, all connected firms must already be counter-parties.

Proposition 2 is a necessary condition for bilateral stability. It neither guarantees existence nor pins down the size of the cliques.

**Theorem 1.** *There exist two numbers $\underline{d}$,$\overline{d}$, and a decreasing function $\phi$ with $\phi(\underline{d}) = \underline{d}$ such that $G = (N, E)$ is bilaterally stable if and only if $G$ consists of disjoint cliques and*

1. *either all cliques have orders between $\underline{d} + 1$ and $\overline{d} + 1$,*
2. *or only one clique has order less than $\underline{d}$, say $\underline{d}_G + 1 \leq \underline{d}$, and all other cliques have orders between $\phi(\underline{d}_G) + 1$ and $\overline{d} + 1$.*

*As for existence, there exists $\underline{\alpha} < 1$ such that for all $\alpha > \underline{\alpha}$, there is an $\underline{n}$ such that for all $n > \underline{n}$, there exists a bilaterally stable network.*[7]

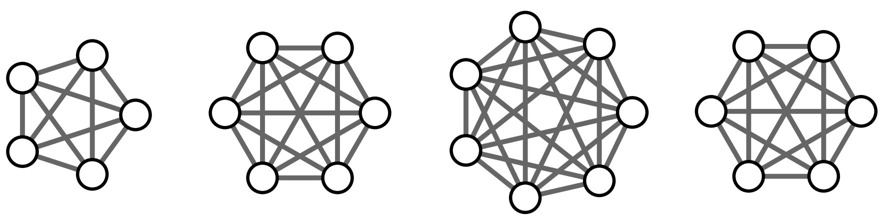


Figure 2: Structure of the periphery under bilateral stability

There can be many bilaterally stable networks because such networks are immune to deviations by pairs only. For bilateral stability, firms only need to have bounds on their payoffs and default risks relative to each other in order to discipline their marginal gains from adding one more link. To make a sharper prediction, we study a refinement of bilateral stability: group stability. Group stability gives uniqueness (up-to labeling), with the caveat that existence can be hindered by cycles of deviations caused by integrality problems.

[7] $\underline{\alpha} < \alpha$ is needed to ensure $\underline{d}$ and $\overline{d}$ are not equal. $n \geq \underline{n}$ is needed to ensure that orders of the cliques can be arranged between $\underline{d}$ and $\overline{d}$ in way to sum up to $n$. Indeed, for small $\alpha$, if $n$ is odd, $\underline{d} = \overline{d} = 2$ and so the smallest clique has to be singleton, but $\phi(1) > 1$. So there does not exist a bilaterally stable network in that case. If $n$ and $\alpha$ are not too small, the orders of cliques in a bilaterally stable network can be arranged in a way to sum up to $n$ and we have existence.

### 2.2.3 Group stable networks

A candidate group stable network consists of disjoint cliques. The expected payoff of a firm in a $(d+1)$-clique firms is

$$V(d) = U(d, (0.5)d(d+1)) = d\alpha^{d(d+1)}\theta.$$

Let the optimal degree $d^* = argmax_{d \leq \tilde{d}} V(d)$. Note that $V$ is strictly log-concave (so single-peaked) and $d^*$ is generically well defined.

**Proposition 3.** *A group stable network must consist of a collection of disjoint cliques, all but at most one of order $d^*+1$. The remaining clique must be of order at most $d^*+1$.*

A network that is group stable necessarily consists of a collection of disjoint cliques of order $d^*+1$ and possibly one 'left-over' clique of a smaller order. This is potentially not group stable. To see why, assume that the left-over clique is of order 1. This single left-over firm would prefer any number of links to none, and so it will be willing to form links with some other firms in a clique. Firms that link with the left-over firm benefit because the left-over firm is not exposed to other counter-parties; therefore, there is no excess risk of contagion from adding a link with the left-over firm. This would constitute a profitable deviation. Now that the component is not a clique, by the argument that underlies Proposition 2, one of the firms which had not linked with the left-over firm would cut an existing incident link and link to the left-over firm. The left-over firm would do the same. Both will improve their payoffs by reducing the risk in the component. Such deviations would keep materializing one after the other, until firms in the original clique of order $d^*+1$ will have lost many links among themselves, and they would rather jointly deviate back to their original disjoint clique of order $d^*+1$, and cut all links with the left-over firm. Such cycles of deviations can be ruled out by farsighted stability. To avoid complications, we make a parity assumption about $n$.[8]

**Theorem 2.** *Suppose that $n$ is divisible by $d^*+1$. There exists a unique (up-to permutation) group stable network and it consists of disjoint $(d^*+1)$-cliques.*

The network formed is illustrated in Figure 3. Now that all firms belong to a clique of order $d^*+1$, the possible profitable deviations would have to match this optimal payoff for all deviators at the expense of non-deviators. The proof shows that the deviator with the smallest degree after the deviation cannot improve its payoff because, after a deviation, non-deviators in a clique still have all the links among themselves, and the other deviators have

[8]This 'integrality' problem does not arise with bilaterally stable networks because there is some flexibility in the orders of the cliques for bilateral stability.

more links than the smallest-degree deviator. Accordingly, after the deviation, there are too many links in the component of the smallest-degree deviator compared to its degree and its payoff cannot exceed $V(d^*)$.

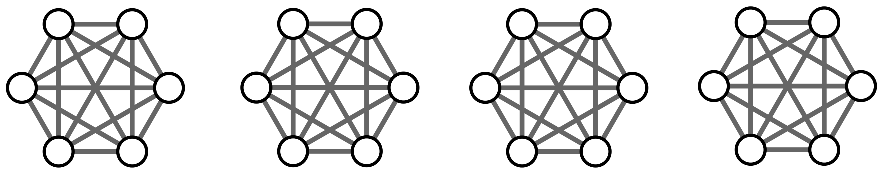


Figure 3: Structure of the periphery under group stability

#### 2.2.4 Efficient networks

Group stable networks are Pareto efficient in general, but there is typically a tradeoff between stability and utilitarian efficiency, as argued in Jackson and Wolinsky (1996). This is not so here. Call a realized network (utilitarian) *efficient* if it maximizes the sum of expected payoffs of firms among all realized networks. Consider a connected component with $e$ links. A firm in the component with degree $d$ enjoys an expected payoff of $d\alpha^{2e}\theta$. Therefore, the sum of payoffs of firms within the component is $2e\alpha^{2e}\theta$. It follows then, that the problem of finding an efficient network devolves into two parts: how to partition firms into components and how many links to put into each component.

**Theorem 3.** *Suppose that $n$ is divisible by $d^* + 1$. A network is efficient if and only if it is group stable.*

There are two reasons a clique or order $d^* + 1$ is the unique efficient network. Only the number of links in a component matters for the efficiency of the component. Therefore, conditional on a certain number of links $e$, the efficient configuration partitions the vertices into as many components as possible, with each having exactly $e$ links. That is, components must be as dense as possible. This is why the efficient configuration features disjoint cliques. Conditional on disjoint cliques, the efficient size of a clique should maximize the average payoff of firms in the clique. Average payoff in a disjoint clique with $d + 1$ firms is $d\alpha^{d(d+1)}$, and so the efficient order of the clique is $d^* + 1$. Note that, all bilaterally stable networks other than the group stable network are inefficient.

Note that group stability does not imply efficiency in general. Examples are easy to construct. Group stability implies Pareto efficiency though. Moreover, under transfers or various sharing rules, strong stability can imply efficiency depending on the sharing rule.

**Remark:** Jackson and Wolinsky (1996) find a cleavage between pairwise stability and efficiency. Pairwise stability does not allow for deviations in which a pair simultaneously adds their missing link and deletes some of their existing links. Accordingly, pairwise stability is weaker than bilateral stability. Allowing a deviation to simultaneously add and delete links is critical in showing that bilaterally stable networks must consist of cliques. The clique structure turns out to be essential in the analysis of efficiency. Blume et al. (2013) also find that their pairwise stable networks are not efficient. Their notion of efficient is a worst-case one, very different from the one employed here. Farboodi (2015) also finds that formed networks are inefficient, despite having group stability as the solution concept.

## 2.3 Peripheral Systemic Risk and the Volatility Paradox

### 2.3.1 Peripheral systemic risk

Recall that we are presenting the model from outside-in. The disjoint clique structure will be the structure of the periphery within the general model. Accordingly, the systemic risk that we study in this section is actually the peripheral systemic risk of the general model, i.e. the risk that most peripheral firms default because of contagion initiated by shocks to the periphery firms. Later, when we study the full model of the core-periphery, we will also be interested in central systemic risk, i.e. the risk that most peripheral firms default because of contagion initiated by shocks to the core.

The network (periphery) formed consists of many disjoint cliques. Each clique is subject to contagion within itself. There is an $\alpha^{d^*(d^*+1)}$ probability that all shocks to a clique are good, therefore no firm in the clique defaults. Otherwise, all firms in the clique default. Since cliques are disjoint, the defaults of cliques are independent events. This allows us to pin down the exact distribution of the number of defaults, which is reported in Appendix B. Figure 4 illustrates the distribution of the fraction of firms that default.

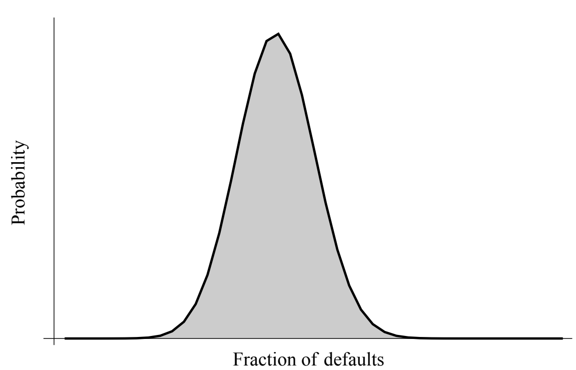



Figure 4: Probability distribution of the fraction of defaults

Systemic risk is typically viewed as the tail event that a large fraction of the system is damaged. For simplicity, we study the extreme tail event that *all* peripheral firms default. Other measures of systemic risk are studied in Appendix B. The probability that all firms in a disjoint clique default is $1-\alpha^{d^*(d^*+1)}$. Since there are $\frac{n}{d^*+1}$ many components, systemic risk (in the absence of the core) is

$$\left(1-\alpha^{d^*(d^*+1)}\right)^{\frac{n}{d^*+1}}. \tag{3}$$

Of interest is how systemic risk changes in response to $\alpha$, which is the model parameter that captures the exogenous and fundamental risk in the economy. On one hand, for a fixed network, as $\alpha$ increases, projects become less risky, contagion is triggered less often, and so the economy is in a fundamentally better state. This reduces any reasonable measure of systemic risk for a fixed network. On the other hand, for fixed $\alpha$, more interconnected networks create more contagion in our model. This increases systemic risk. In the endogenous network that is formed, as $\alpha$ increases, $d^*$ increases, and the network becomes more interconnected. This is depicted by the solid line in Figure 5*a*. A priori it is unclear whether the reduction of risk due to higher $\alpha$ or the increment in risk due to higher interconnectedness dominates.

In Figure 5*b*, the solid line is the systemic risk. Systemic risk of the group stable and efficient network increases with $\alpha$ (modulo the non-monotonicity due to discreteness of the model). We call this the *peripheral volatility paradox*, which is the subject of the next section.

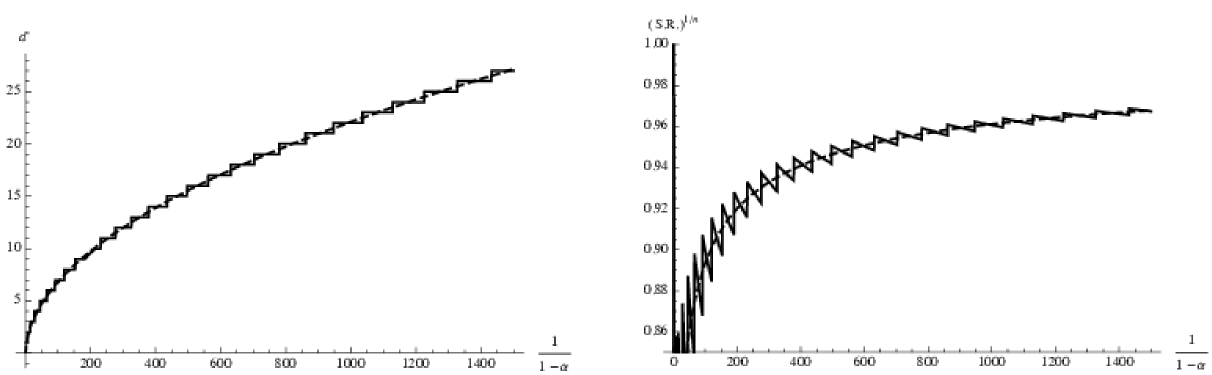



Figure 5*a*: Degree $d^*$ in $\alpha$ Figure 5*b*: Systemic risk in $\alpha$

Figure 5: Degree $d^*$ and systemic risk

### 2.3.2 Peripheral volatility paradox

To understand why systemic risk increases in $\alpha$ consider a "smoothed" version of the expected payoff of a firm $V_S(x) = x\alpha^{x^2}$. The optimal $x$ for $V_S$ is given by $\sqrt{(-2\ln(\alpha))^{-1}}$ which is the dashed line in Figure 5$a$.[9] This makes $\alpha^{x^2} = \mathbf{e}^{-0.5}$ where $\mathbf{e}$ is Euler's constant. Then, roughly, the probability of every firm within a clique not defaulting is $\mathbf{e}^{-0.5}$ which is independent of $\alpha$. The corresponding estimate for systemic risk is then

$$\left(1 - \boldsymbol{e}^{-0.5}\right)^{n\sqrt{-2\ln[\alpha]}}$$

which is the dashed line Figure 5$b$. Note that this is increasing in $\alpha$.

It is striking that the risk that a clique fails is (roughly) constant in $\alpha$, but systemic risk increases particularly as the number of firms $n$ is fixed. Notice that if one increases $n$, systemic risk actually *decreases* because the probability that more cliques fail is smaller than the probability that few cliques fail, given that the failures of cliques are independent events with probability $1 - \boldsymbol{e}^{-0.5}$.

Systemic risk increases because of the *endogenous correlation of default risk* that stems from network formation. For fixed $n$, as $\alpha$ increases, clique size grows, and so the number of cliques decreases. There are a smaller number of cliques, each of which has a constant probability of failing. So, the probability that all cliques fail *increases.* In other words, more firms become highly correlated with an increase in $\alpha$ and the heightened correlation increases the probability of the tail event, i.e. systemic risk. This yields a network version of the volatility paradox propounded by Brunnermeier and Sannikov (2014).[10]

### 2.3.3 Weak contagion: Network externalities cause peripheral volatility paradox

The volatility paradox is not a consequence of the strength of contagion. To see why, relax Assumption 1 and allow for small $\beta$ while keeping $\gamma$ large (only within Section 2.3.3). That is, one bad shock is still sufficient to force a firm into default, but now many counter-parties must fail in order to force a firm into default. For example, for $\theta = 1$ and $\beta = 2$, a firm with good shocks defaults only if more than half its counter-parties default. Accordingly, in a disjoint clique, if more than half the firms suffer bad shocks, all firms default.

We focus on networks that consist of disjoint cliques that maximize the average payoff of firms in the clique.[11] When $\beta$ is small, the payoff of a firm in a $(d+1)$-clique is not $d\alpha^{d(d+1)}\theta$

[9] It is easy to show that $\left| d^* - \left( \sqrt{(-2\ln(\alpha))^{-1} + \frac{1}{16}} - \frac{1}{4} \right) \right| < 1$ for any $\alpha$.

[10] Recall that there is an upper bound $\tilde{d} \leq n-1$ on the number of links a firm can have. Once $\alpha$ becomes too large and hits $\left(\frac{\tilde{d}-1}{\tilde{d}}\right)^{\frac{1}{2\tilde{d}}}$, $d^*$ becomes $\tilde{d}$ and the cliques cannot get any larger. Systemic risk cannot get any larger and starts decreasing. $\tilde{d}$ could very well be equal to $n-1$ meaning that the cliques would grow until becoming the complete network.

[11] We show in Appendix C that this particular network is bilaterally stable if $\alpha^2 + \frac{\theta}{\beta} < 1$.

anymore. Denote by $\mathbb{P}_{bin}$ and $\mathbb{F}_{bin}$the binomial PDF and CDF. The expected payoff of a firm in a $(d+1)$-clique becomes

$$V(d) = \alpha^d \times \sum_{f=0}^{\left\lfloor \frac{d\theta}{\beta} \right\rfloor} \mathbb{P}_{bin}\left[f; d, 1-\alpha^d\right](d\theta - \beta f)$$

$$= d\alpha^d \times \left\{ \theta \mathbb{F}_{bin}\left[\left\lfloor \frac{d\theta}{\beta} \right\rfloor; d, 1-\alpha^d\right] - \beta\left(1-\alpha^d\right)\mathbb{F}_{bin}\left[\left\lfloor \frac{d\theta}{\beta} \right\rfloor - 1; d-1, 1-\alpha^d\right]\right\}. \quad (4)$$

Absent a closed-form solution to $d^*$ we resort to simulations for comparative statics. Consider $\theta = 1$ and $\beta = 1.5$. $d^*$ increases in $\alpha$. Figure 6 plots the corresponding systemic risk. Systemic risk increases as the probability of good shock $\alpha$ increases, even under mild network externalities. Therefore, the volatility paradox is not just a consequence of the strength of contagion.

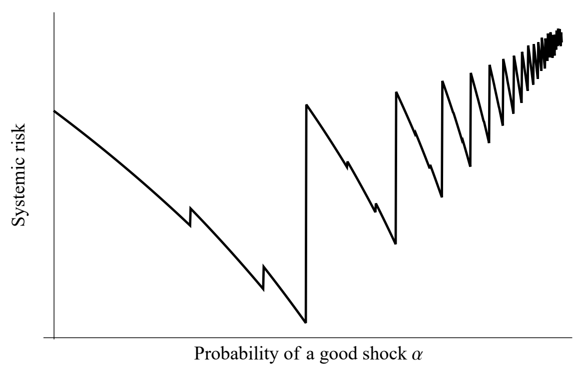



Figure 6: Peripheral volatility paradox persists under weaker contagion: $\theta = 1$, $\beta = 1.5$

This does not mean that volatility paradox is independent of contagion. To see why, consider the case of $\beta < 1$. Then, there is no contagion and the only risk that a firm faces is the immediate shocks to its links. A firm's payoff is $d\alpha^d\theta$. Note that $argmax\, d\alpha^d = \left\lfloor \frac{1}{1-\alpha} \right\rfloor$, which is approximately $(-ln(\alpha))^{-1}$ for large $\alpha$. The default risk of a firm is approximately $1 - \mathbf{e}^{-1}$. Systemic risk is $(1 - \mathbf{e}^{-1})^n$, which is constant. This illustrates that in our model the volatility paradox arises due to network externalities. Whether the contagion is weak or strong, the volatility paradox is present. If there is no contagion, there is no volatility paradox.

### 2.3.4 Inferring systemic risk from a dense periphery

Earlier we drew attention to a debate about whether interconnectedness of firms is a significant contributor to systemic risk. An alternative theory is that the risk faced is systematic, i.e. popcorn. We model the popcorn story as perfect correlation in the link shocks. Suppose that there is probability $\sigma$ that all links are good, and probability $1-\sigma$ that all links are bad. Now there is no risk of contagion and all risk is due to common exposures.

**Proposition 4.** *Suppose that $n$ is divisible by $\tilde{d}+1$. Under 'popcorn', the unique group stable (and the unique bilaterally stable) network consists of disjoint cliques of order $\tilde{d}+1$.*

When firms face systematic risk (popcorn) rather than systemic risk (dominoes), they form highly interconnected networks in order to reap the benefits of trade. Under popcorn, all firms are in the same boat and the major risk is due to common exposure rather than contagion. There is no need to refrain from forming too many links. If the underlying common shock is good, then firms get higher returns by having more links. If the underlying common shock is bad, eliminating some links cannot change the outcome.

Distinction between dominoes and popcorn is related to underestimating systemic risk. What should a policy maker infer from a highly interconnected system about tail risk in the economy? The critical observation is that under popcorn, a highly interconnected network is uninformative about the probability of system-wide failure. High interconnectedness in an endogenous network is informative about the probability of system-wide failure only if there is a risk of contagion. We formalize this below.

Suppose an outside observer is uncertain about whether he is facing popcorn or dominoes. If the realized network consists of $(d+1)$-cliques where $d<\tilde{d}$, it can be inferred that the risk is dominoes. $\alpha$ and the systemic risk can be inferred from $d$. In particular, systemic risk is given by Equation (3), which is roughly $(1-e^{-0.5})^{\frac{n}{d+1}}$. On the other hand, if the realized network consists of $\left(\tilde{d}+1\right)$-cliques, the network is uninformative about whether the risk is popcorn or dominoes.

In the popcorn world, systemic risk is $1-\sigma$, which can take any arbitrarily large value between 0 and 1. In the dominoes world, systemic risk is $1-\alpha^{\tilde{d}\left(\tilde{d}+1\right)}$ which is bounded away from 1.

**Theorem 4.** *In the popcorn world, observing an endogenously formed dense network, i.e. $\left(\tilde{d}+1\right)$-cliques, is uninformative: the probability of system-wide failure can range from $0$ to $1$. In the dominoes world, observing an endogenously formed dense network is informative about risk: the probability of system wide failure can range from $0$ to*

$$\left(1-\left(\frac{\tilde{d}-1}{\tilde{d}}\right)^{\frac{\tilde{d}+1}{2}}\right)^{\frac{n}{\tilde{d}+1}}.$$

In the dominoes world, firms never form a network which has a very high systemic risk. It is easier to notice this by recalling that the probability that a clique defaults in the endogenous network is roughly equal to $1 - \boldsymbol{e}^{-0.5}$. Accordingly, in the endogenous network, systemic risk is always bounded away from 1. As $\alpha$ grows larger and $d^*$ hits $\tilde{d}$, systemic risk gets even smaller. Accordingly, upon observing a complete network, the outside observer can conclude that systemic risk can range from 0 to the upper bound.

Note that the upper bound is smaller than 0.5 for all $n \geq 6$ and $\tilde{d} \leq n - 1$. Therefore, an outside observer who observes an endogenously formed dense network and believes it is the dominoes world would think that systemic risk is smaller than 0.5, whereas under the popcorn world, systemic risk can be arbitrarily close to 1. The same result holds for all measures of systemic risk we consider in Appendix B.

In sum, upon observing a dense network but not the correlation structure, one cannot accurately upper bound the probability of system-wide failure. Mistakenly believing that the risk is systemic rather than systematic can lead to underestimating the probability of system-wide failure.

# 3 The Core

## 3.1 The Model of the Core-Periphery

Here we extend the model in two dimensions. First, we introduce node shocks: shocks that directly impact firms rather than links. They model idiosyncratic operational firm costs. Contagion can be triggered by node shocks, which introduces another source of systemic risk. Second, we introduce contagion-resilient types: a type of firm whose main risk of defaults stems from node shocks. This captures large and experienced firms that face smaller interest rates from lenders. As before, we first introduce the simplified model in Section 3.1.1, then make the interpretation concrete in Section 3.1.2.

### 3.1.1 Model

There are two types of firms: *contagion-resilient* types and *normal* types. There are $m$ contagion-resilient firms denoted $M = \{1, ..., m\}$ and $n$ normal firms denoted $N = \{m + 1, ..., m + n\}$. All pairs of firms can form links. As before, we assume that firms form stable networks given the expectation of their payoffs in the continuation. After the network is formed, links receive good or bad shocks. Moreover, each firm $i$ receives a *node shock* which is good or bad. A firm with a bad node shock defaults and earns 0.

For normal firms, the probability of a good link shock is $\alpha$ and a good node shock is $\zeta$. A normal firm $i \in N$ with $d_i$ counter-parties, $f_i$ many defaulting counter-parties, and $b_i$ many bad links that receives a good node shock, has payoff $u_i = d_i\theta - \beta f_i - \gamma b_i$ if it continues. If $u_i \geq 0$, $i$ continues and enjoys $u_i$. Otherwise $i$ defaults and earns 0. Assumption 1 still holds for normal firms.

For contagion-resilient firms, the probability of a good link shock is $\alpha'$ and a good node shock is $\zeta'$. A contagion-resilient firm $i \in M$ who has received a good node shock, with $d_i$ many counter-parties, $f_i$ many defaulting counter-parties, and $b_i$ many bad links has payoff $u'_i = d_i\theta' - \beta' f_i - \gamma' b_i$ if it continues. If $u'_i \geq 0$, $i$ continues and gets $u'_i$. Otherwise $i$ defaults and gets 0. As before, the outcome of contagion corresponds to the cooperating equilibrium. The difference between contagion-resilient and normal firms is encapsulated in the following assumption.

**Assumption 2.** *Contagion-resilient firms always benefit from each link: $\theta' > \gamma' + \beta'$.*

Assumption 2 implies that contagion-resilient firms do not transmit contagion. While they incur losses from link shocks and counter-party failures, such losses never make a contagion-resilient firm default. Nevertheless, contagion-resilient firms can initiate contagion with a bad node shock, and so they still pose a threat to the system. In fact, the model allows $\zeta' < \zeta$ so that contagion-resilient firms can be even more vulnerable to node shocks than normal firms. In the rest of the paper, contagion-resilient firms will simply be called *resilient* firms, but the reader should bear in mind that such firms can spark contagion more often than normal firms.

Under Assumption 2 we can pin down the expected payoffs of firms in a given network as follows. Call a path on the network a *normal path* if it consists of normal firms only.[12] For $i \in N$, consider the maximal rooted subtree rooted at $i$ consisting of all normal firms that can be reached by normal paths from $i$. Denote by $d_i^N$ the number of normal firms in this subtree. Denote by $d_i^M$ the number of resilient firms that are counter-parties of the $d_i^N$ normal firms in the subtree. Let $e_i^N$ be the number links in the entire subnetwork of these $d_i^N$ firms. Denote by $e_i^M$ the number of links in the cutset that separates the subtree from the rest of the network.

Note that the probability that a normal firm $i$ does not default is given by

$$p_i := \zeta^{d_i^N} \zeta'^{d_i^M} \alpha^{2e_i^N + e_i^M}. \tag{5}$$

[12] The trivial path from $i \in N$ to itself is also a normal path.

**Proposition 5.** *The expected payoff of $i \in N$ is*

$$u_i = d_i \theta p_i.$$

*The expected payoff of $i \in M$ is*

$$u_i = d_i \zeta' \left(\theta' - \alpha'\gamma' - \zeta'\beta'\right) - \left( \sum_{j \in N \cap D_i} \left( p_j - \zeta'^2 \right) \right) \beta'.$$

**Remark:** We use different parameter values between normal and resilient firms in order to make the equations easier to interpret. For example, if we were to take $\zeta = \zeta'$, the formula for $p_i$ in (5) would involve a term $\zeta^{d_i^N + d_i^M}$ instead of $\zeta^{d_i^N} \zeta'^{d_i^M}$. We find the latter easier to understand and interpret. In fact, the only parameter difference between the two types of firms needed for our results is that $\theta \neq \theta'$. All other parameters can be taken to be equal: $\gamma = \gamma'$, $\beta = \beta'$, $\alpha = \alpha'$, $\zeta = \zeta'$.

### 3.1.2 Meaning of the model

Following up on Section 2.1.2, we describe the node shocks and the main difference between normal and resilient firms with regards to joint ventures. This section is self-contained and can be omitted without loss of comprehension. All assumptions and propositions related to this section appear with the prefix 'M'.

Absent node shocks, normal firms are identical to the firms of Section 2.1.2. Suppose that firms have to incur an operational cost to continue their business, irrespective of their investments and project management costs. For each normal firm $i$, there is a probability $1 - \zeta$ that $i$ has a high operational cost $K$ to continue, and there is a $\zeta$ probability that $i$ has an operational cost of 0.

Resilient firms correspond to firms that enjoy low interest rates, hence high profit margins from their projects. Define all parameters analogously for resilient firms: $L'$ is the credit line, $R'$ interest promised to lenders. (We endogenize $R'$ later in Section 4. All results go through with endogenous $R'$ as well.) $P'$ is the return per unit investment into the projects, $\overline{C}' > \underline{C}'$ are management costs, $K'$ is the high operational cost, $\overline{B}' > \underline{B}'$ are returns from projects, $\alpha'$ is the probability of having low management cost for a project, and $\zeta'$ is the probability of having a low operational cost.

We maintain Assumption M 1 and Assumption M 2 throughout Section 3.1.2. Regarding node shocks and resilient firms, we make some more assumptions.

**Assumption M 3.** *High operational costs are large enough to force firms into default:* $KP > L\left(\overline{B} - \underline{C} - PR\right) > 0$ *and* $K'P' > L'\left(\overline{B}' - \underline{C}' - P'R'\right) > 0$*.*

Assumption M 1 and Assumption M 3 give us the counterpart of Proposition 1 that accounts for operational costs. Define $\theta'$, $\beta'$, $\gamma'$ for resilient firms analogously with normal firms as in Equation (2):

$$\theta' = \overline{B}' - \underline{C}' - P'R',\ \beta' = \overline{B}' - \underline{B}',\ \gamma' = \overline{C}' - \underline{C}'.$$

**Proposition M 2.** *Firms with high operational costs drop all projects. The best response of a firm* $i$ *with low operational costs is to either keep all projects or to drop all projects. If* $i$ *drops all projects, its payoff is* 0*. If* $i$ *keeps all projects, its payoff is* $d_i\theta - f_i\beta - b_i\gamma$ *if* $i \in N$ *and* $d_i\theta' - f_i\beta' - b_i\gamma'$ *if* $i \in M$*.*

Assumption M 2 still holds for normal firms. Since margins are thin and operational costs are large, for $i \in N$, if any link is bad, or if any counter-party defaults, $i$ defaults. This does not suffice to determine the expected payoff of a normal firm in a given network because we need to pin down the behavior of resilient firms as well.

**Assumption M 4.** *Resilient firms have low interest rates:* $\underline{B}' - \overline{C}' > P'R'$*.*

Assumption M 4 is the counterpart of Assumption 2. Under Assumption M 4, a resilient firm with a low operational shock always continues, irrespective of shocks to its links and the defaults by its counter-parties. Resilient firms do not transmit contagion, but they can trigger contagion with bad operational shocks, and so they still pose a threat to the system.

**Remark:** Recall the remark at the end of Section 3.1.1 where we mentioned that the only difference needed in parameter values is that $\theta \neq \theta'$. In other words, all that is necessary four our following results is that $R \neq R'$. All other parameters can be taken to be identical across normal and resilient firms: $K = K'$, $P = P'$, $L = L'$, etc.

## 3.2 Stability and Efficiency of the Core-Periphery

The final piece needed to determine the group stable network is that resilient firms do not have significantly more idiosyncratic risk than normal firms, and so resilient firms are preferred counter-parties with respect to the probability of triggering contagion.

**Assumption 3.** *Resilient firms' node risk is not too high compared to normal firms:* $\zeta' > \alpha\zeta$*.*

This does not preclude $\zeta > \zeta'$, so resilient firms can actually be *riskier* in terms of node shocks. If it were the case that $\zeta' = \zeta$, this assumption would be redundant.

### 3.2.1 Bilaterally stable networks

When a normal firm $i$ forms a link with a normal firm $j$, it must consider three risks aside from contagion through $j$. First, $j$ can suffer a bad node shock with probability $1-\zeta$. Second, $j$ can suffer a bad link shock at link $\{i, j\}$ with probability $1-\alpha$. Third, $i$ can suffer a bad link shock at link $\{i, j\}$ which has probability $1-\alpha$.

If $i$ forms a link with a resilient firm $k$, it must consider only two such risks. First, $k$ can suffer a bad node shock with probability $1-\zeta'$. Second, $i$ can suffer a bad link shock at link $\{i, k\}$ with probability $1-\alpha$. While $k$ can suffer a bad link shock at link $\{i, k\}$ with probability $1-\alpha'$, this does not make $k$ default, so it is not a risk for $i$.

Since $\zeta'\alpha > \zeta\alpha^2$, resilient firms are preferred counter-parties, although it is possible that $\zeta' < \zeta$. This leads to the following result on bilaterally stable networks. Set $d_M^* = argmax_{d \leq \tilde{d}} d\left(\zeta'\alpha\right)^d$. It is easy to see that $d_M^* = \left\lfloor \frac{1}{1-\alpha\zeta'} \right\rfloor$.

**Theorem 5.** *(Case of small core) If $d_M^* > m$, any bilaterally stable network must be a core-periphery that satisfies the following:*

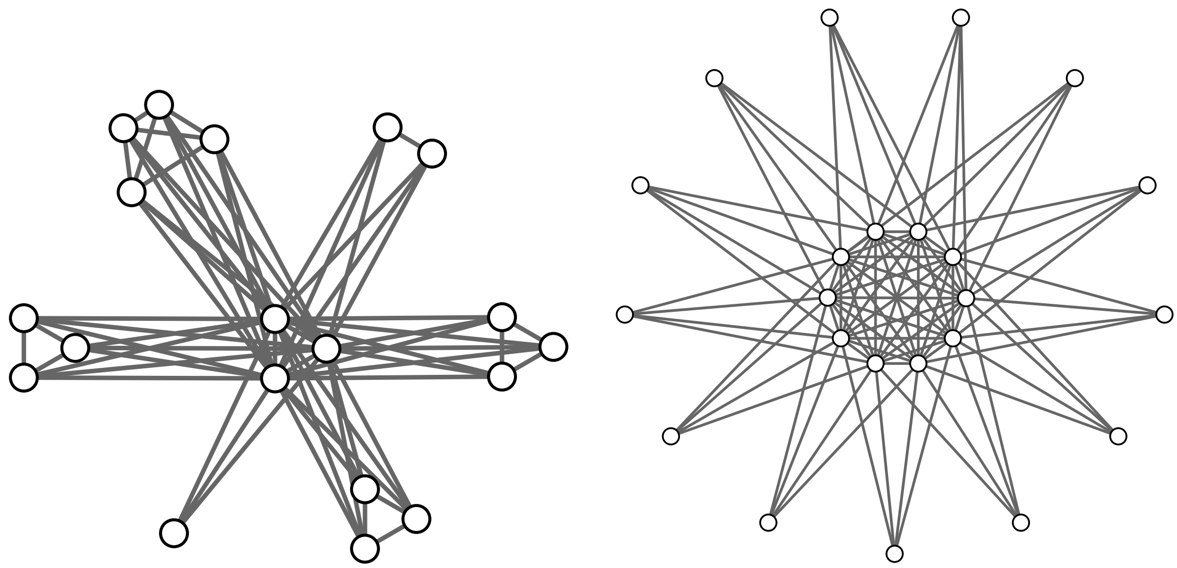
1. *(Core) Each resilient firm is counter-parties with all resilient firms and all normal firms.*

2. *(Periphery) Each normal firm is counter-parties with all resilient firms and some normal firms. Excluding resilient firms and their incident links, normal firms form disjoint cliques among themselves.*

*(Case of large core) If $d_M^* \leq m$, any bilaterally stable network must be a core-periphery that satisfies the following:*

1. *(Core) Each resilient firm is counter-parties with all resilient firms and some normal firms.*

2. *(Periphery) Each normal firm has $d_M^*$ resilient counter-parties and no normal counter-parties.*



Small core | Large core

Figure 7: Structure of the bilaterally stable networks

#### 3.2.2 Group stable networks

Set $d_N^* = argmax_{d\geq 0}\,(m+d)\,(\alpha^m\zeta)^d\,\alpha^{d(d+1)}$. Note that if $d_M^* \leq m$ then $d_N^* = 0$.

**Theorem 6.** *Suppose that $n$ is divisible by $d_N^*+1$.*

*(Case of small core) If $d_M^* > m$, a network is group stable if and only if it is a core-periphery of the following form:*

1. *(Core) Each resilient firm is counter-parties with all resilient firms and all normal firms.*
2. *(Periphery) Each normal firm is counter-parties with all resilient firm and some normal firms. Excluding resilient firms and their incident links, normal firms form disjoint cliques of order $d_N^*+1$ among themselves.*

*(Case of large core) If $d_M^* \leq m$, a network is group stable if and only if it is a core-periphery of the following form:*

1. *(Core) Each resilient firm is counter-parties with all resilient firms and some normal firms.*
2. *(Periphery) Each normal firm has $d_M^*$ resilient counter-parties and no normal counter-parties.*

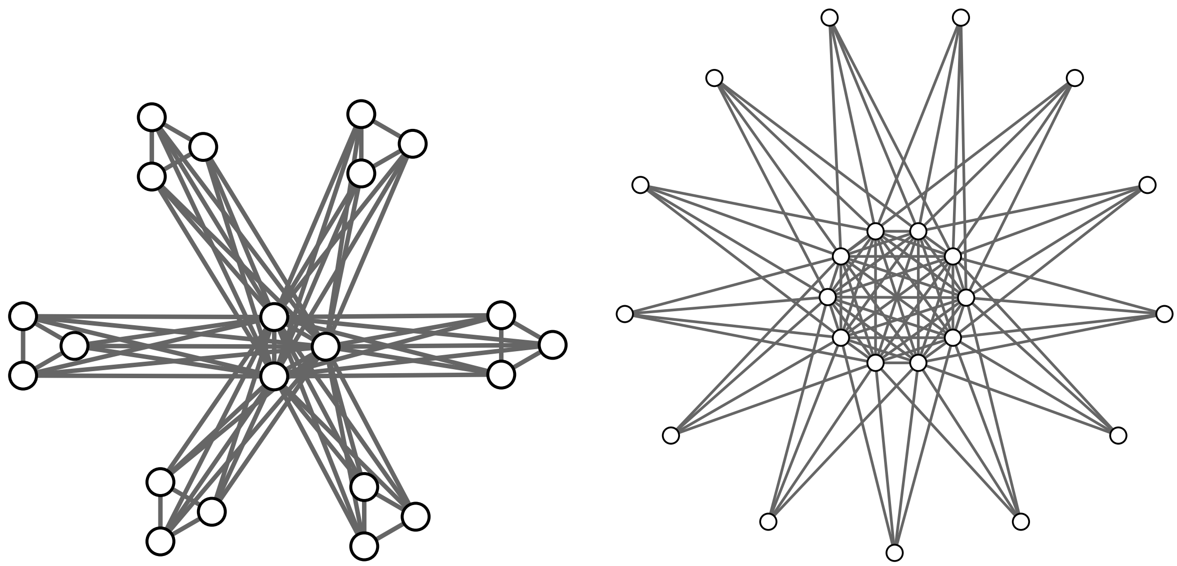



Figure 8: Structure of group stable networks

The possible group stable networks are displayed in Figure 8. Resilient firms are preferred counter-parties because Assumption 2 makes the only risk that a resilient firm imposes on its counterparties is its node shock, and given that this is the case, $\zeta' > \alpha\zeta$ makes resilient firms node shock less of a risk then a normal firm's combined risk from the node shocks and the link shock. This does not mean that linking with the resilient firms entails no risk. The group stability of the cliques in the periphery is a consequence of the results we have established earlier regarding the periphery. The group stability of the links to the core is a consequence of resilient firms being preferred counterparties, despite the fact that connecting to the core still entails counterparty risk.

### 3.2.3 Efficient networks

**Theorem 7.** *Suppose $n$ is divisible by $d^*_N + 1$.*

*(Case of small core) If $m < d^*_M$, the unique efficient network is group stable.*

*(Case of large core) There exists $\underline{m} > d^*_M$ such that if $m > \underline{m}$, the unique efficient network is the core-periphery in which*

1. *each resilient firm is counter-parties with all resilient firm and all normal firms,*

2. *and each normal firm is counter-parties with all resilient firms and no normal firms.*

When resilient firms are scarce, it is efficient to connect each normal firm to every resilient firm. For normal firms, this configuration connects them to the least risky firms and also makes the size of cliques smaller in the periphery, minimizing the externality imposed by contagion. We showed earlier that within the periphery, the efficient configuration is the clique structure. As for resilient firms, an extra link is always beneficial to them.

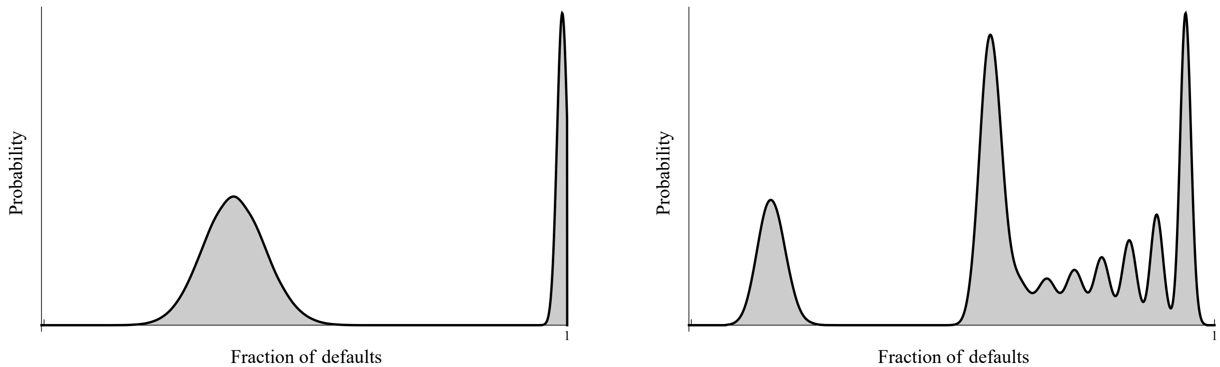
When there is a surplus of resilient firms, normal firms are harmed by having too many resilient counter-parties. This is because resilient firms are subject to node shocks, and so it is possibly inefficient to connect all resilient firms to all normal firms. On the other hand, the benefit enjoyed by resilient firms may exceed the loss suffered by normal firms. If there are sufficiently many resilient firms relative to normal firms, even at the expense of reducing the payoffs of normal firms to zero, the positive gains obtained by the resilient firms exceeds the loss suffered by the normal firms.

## 3.3 Core-periphery and the Central Systemic Risk

### 3.3.1 Central systemic risk

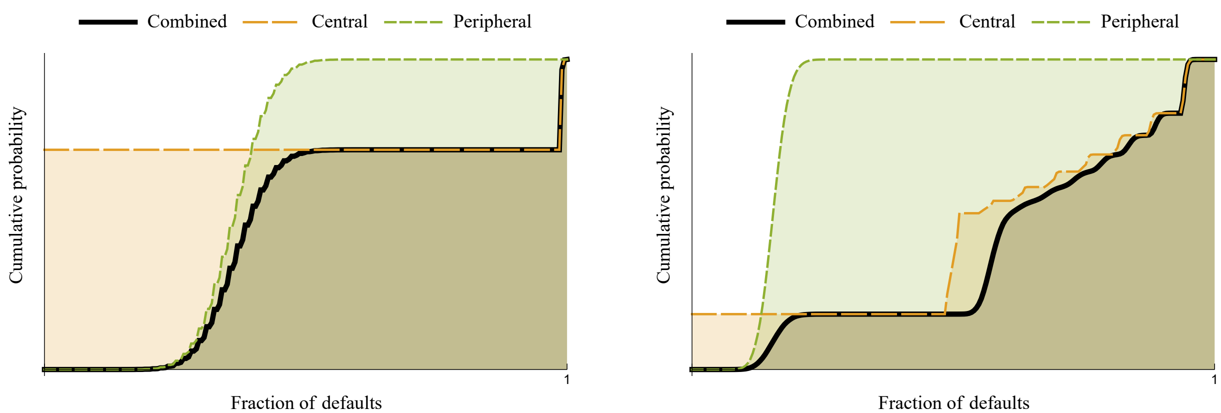
Linking to the core consisting of resilient firms is not risk free. While the core prevents contagion spreading from one component in the periphery to another, resilient firms are still subject to bad node shocks. For example, when $d_M^* \geq m$, if a resilient firm at the core suffers a bad node shock, it drives *the entire periphery* into default. Thus, the core is the major source of systemic risk in the economy. We call the probability of system-wide failure due to shocks to firms in the core *central systemic risk*. There is still peripheral systemic risk in the economy, which is the probability of the event that all cliques of peripheral nodes fail.

Figure 9 illustrates the probability distribution of the number of defaults. The tail event in which most firms fail has high probability because if a resilient firm in the core suffers a bad node shock and defaults, it drags many normal firms in the periphery into default. Normal firms' risk of default becomes highly correlated through the node shocks of the core.




Small core Large core

Figure 9: Probability distribution of the number of defaults




Small core Large core

Figure 10: Cumulative probability distribution of the number of defaults with respect to sources of systemic risk

A closed-form expression for the PDF of the distribution of the number of defaults can also be determined. We have shown how this can be done for the periphery in Appendix B. One can incorporate the core similarly. We skip this for brevity and focus on the tail event that all firms in the periphery fail, i.e. systemic risk. In the case of a small core, all peripheral firms have links with all core firms. One bad node-shock to the core causes all peripheral firms into default. If all core firms enjoy good node-shocks, all peripheral firms default only if all cliques in the periphery get bad shocks of their own. The systemic risk is then given by

$$\underbrace{(1-\zeta'^m)}_{Central\,systemic\,risk} + \zeta'^m \underbrace{\left(1-\zeta^{d_N^*+1}\alpha^{\left(m+d_N^*\right)\left(d_N^*+1\right)}\right)^{\frac{n}{d_N^*+1}}}_{Peripheral\,systemic\,risk}. \tag{6}$$

In the case of a large core, there are multiple core-periphery networks that can be formed.

In each of them, each peripheral firm has $d_M^*$ core counter-parties and no peripheral counter-parties. Systemic risk can take various values depending on the overlap between the set of counter-parties of each peripheral firm.[13] The simplest possibility is that all peripheral firms are counter-parties with the same set of $d_M^*$ core firms. In this case, systemic risk is given by

$$\underbrace{\left(1-\zeta'^{d_M^*}\right)}_{Central\,systemic\,risk}+\zeta'^{d_M^*}\underbrace{\left(1-\zeta\alpha^{d_M^*}\right)^n}_{Peripheral\,systemic\,risk}. \tag{7}$$

In what follows, we refer to (7) when we study systemic risk under the large core. In the next section we study the comparative statics of systemic risk to see whether the volatility paradox is present in the core-periphery network as well.

### 3.3.2 Central volatility paradox

In Section 2.3.2, we identified and discussed a peripheral volatility paradox: as the probability of good link shocks $\alpha$ increase, the probability that all firms in the periphery fail via contagion initiated by shocks to the periphery (i.e. peripheral systemic risk) increases. This was specific to the periphery and cliques that emerge within the periphery. Under the core-periphery structure, similar arguments apply to the cliques within the periphery. As for the core, can we expect a different form of volatility paradox due to the shocks to the core? How does the probability that all firms in the periphery default (i.e. central systemic risk) due to shocks to firms in the core change as the probability of good shocks increase?

Consider the shocks to the links. First, we must understand how the structure of the network and the degree of peripheral firms change as $\alpha$ changes. Recall that resilient firms that occupy the core are preferred counterparties. Start with some small $\alpha$ such that $d_M^* \leq m$. Then peripheral firms want to be counterparties with only core firms. Thus $d_N^* = 0$ and so $d^* = d_M^*$. As $\alpha$ increases, eventually, $d_M^* > m$. At this point, there are not enough core firms, and $d^* = d_M^* + d_N^*$. In fact, $d^*$ can be written compactly as $d^* = min\left\{d_M^*, m\right\} + d_N^*$.[14] The way in which $d^*$ changes as a function of $\alpha$ is portrayed in Figure 11$a$.

Next, we study how systemic risk changes in $\alpha$. Beginning with some small $\alpha$, the periphery is willing to form only a few links with the core: $d_M^* < m$. Recall that $d_M^* = argmax_d d\left(\zeta'\alpha\right)^d$. As $\alpha$ increases, so does $d_M^*$, and so the periphery is willing to form more links with the core.

[13]Two sets $X$ and $Y$ have non-trivial overlap if $(X \backslash Y) \cup (Y \backslash X) \neq \emptyset$. When there are non-trivial overlaps between sets of counter-parties of peripheral firms, no closed form expression for systemic risk exists. Nevertheless, we can pin down a value for systemic risk when there are no non-trivial overlaps between sets of counter-parties of any pair of peripheral firms.

[14]It is worthwhile to note that $d^* = argmax_d d\zeta'^{min\{d,m\}}\zeta^{d-min\{d,m\}}\alpha^{d+(d-min\{d,m\})^2}$.

The risk that is internalized by a peripheral firm from adding one more link comes from both the node risk of the core firm, $1-\zeta'$, and the link risk of the newly added link $1-\alpha$. Peripheral nodes internalize the added default risk, $1-\zeta'\alpha$, and add links accordingly as $\alpha$ increases. The change in $d_M^*$ keeps the risk $(\zeta'\alpha)^{d_M^*}$ constant (modulo the discreteness of the model). But central systemic risk increases with $1-\zeta'$ not $1-\zeta'\alpha$. $d_M^*$ increases, so the central systemic risk $1-(\zeta')^{d_M^*}$ increases. As for combined systemic risk, we need to consider peripheral systemic risk too. In fact, $\alpha^{d_M^*}$ increases as $\alpha$ increases because $(\zeta'\alpha)^{d_M^*}$ is kept constant by the endogenous choice of $d_M^*$. Therefore, the peripheral systemic risk decreases because $1-\alpha^{d_M^*}$ decreases. As for the combined effect, note that the core is a more significant source of correlation in the default risk of the periphery. A Taylor approximation shows that the increase in the central systemic risk dominates the decrease in peripheral systemic risk. This is plotted in Figure 11$b$ for small to medium $\alpha$. Combined systemic risk increases in $\alpha$.

Once $\alpha$ reaches a medium level, $d_M^*$ becomes $m$ and there are no more firms in the core to form links with. The peripheral nodes start to link with each other. $d_N^*$ starts growing. What happens to systemic risk once $\alpha$ is large enough that $d_M^* = m$? Now central systemic risk does not change anymore because there cannot be more connections to the core. Central systemic risk is set at $1-(\zeta')^m$. Recall that $d_N^* = argmax_{d\geq 0}\,(m+d)\,(\alpha^m\zeta)^d\,\alpha^{d(d+1)}$. As $\alpha$ increases, $d_N^*$ grows and the cliques in the periphery get larger. It turns out that the increment in $d_N^*$ offsets the increase in $\alpha$ and peripheral systemic risk increases. In order to see why, note that $\zeta^{d_N^*}\alpha^{2d_N^*\left(d_N^*+m+1\right)}$ is made constant (modulo discreteness) by the endogenous choice of $d_N^*$, whereas the risk of a clique is $1-\zeta^{d_N^*}\alpha^{d_N^*\left(d_N^*+m+1\right)}$. $d_N^*$ increases as $\alpha$ increases. In turn, $\zeta^{d_N^*}$ decreases so that $\alpha^{2d_N^*\left(d_N^*+m+1\right)}$ increases. Then $\alpha^{d_N^*\left(d_N^*+m+1\right)}$ increases while $\zeta^{d_N^*}\alpha^{2d_N^*\left(d_N^*+m+1\right)}$ is constant. Therefore, $\zeta^{d_N^*}\alpha^{d_N^*\left(d_N^*+m+1\right)}$ decreases. Accordingly, the risk of a clique increases as $\alpha$ increases in addition to the increased risk from larger cliques and increasing correlation. Peripheral systemic risk increases in $\alpha$. Peripheral systemic risk is the only source of change in combined systemic risk as $\alpha$ increases because central systemic risk is constant. Therefore, combined systemic risk increases in $\alpha$. This is plotted in Figure 11$b$ for medium to large $\alpha$. These comparative statics are summarized in Table 1.[15]

[15] In fact, the probability that an individual firm fails is also increasing in $\alpha$. Nevertheless, since all defaults in the periphery are highly correlated through the shocks to the core, systemic risk also increases.

| Systemic | w.r.t.: | As $\alpha$ increases (link-shock) | | |
|---|---|---|---|---|
| risk | source: | Central | Peripheral | Combined |
| $d^* \leq m$: | | Increasing | Decreasing | Increasing |
| when $\alpha$ crosses a threshold, $d^*$ crosses $m$ | | | | |
| $d^* > m$: | | Constant | Increasing | Increasing |

Table 1: Central and peripheral systemic risk in the probability of good link-shock $\alpha$

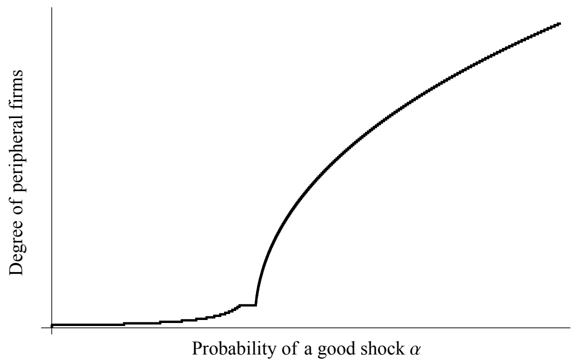


Figure 11$a$: Degree of normal firms in $\alpha$

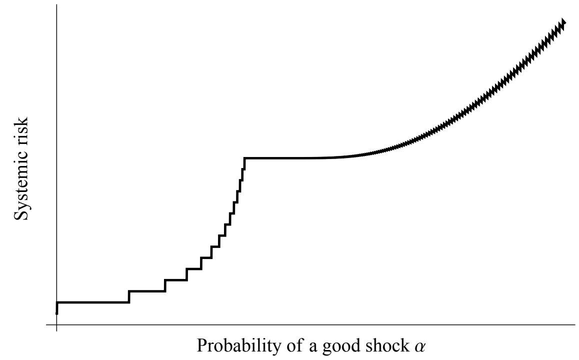


Figure 11$b$: Systemic risk in $\alpha$

Figure 11: Volatility paradox

Changing the probability of good node shocks impacts the network as well. However, the network never switches from $d^* \leq m$ to $d^* > m$ by changing $\zeta$ or $\zeta'$. Whether the network is in the "small core" region or the "large core" region is determined by $\alpha$. Thus we relegate the analysis of the node shocks to Appendix D.

### 3.3.3 Weak contagion: Network externalities cause central volatility paradox

In Section 2.3.3, we momentarily relaxed Assumption 1 and illustrated that the peripheral volatility paradox is not a consequence of the strong contagion. We do the same in this section. We relax Assumption 1 by allowing for small $\beta$ (only for Section 3.3.3) and show that the central volatility paradox persists.

For a small core, peripheral firms connect with each other and there are cliques in the periphery. Therefore, we focus on the large core case in order to single out the central volatility paradox. Consider a network that consists of singletons in the periphery, such that each peripheral firm has links with $d$ core firms. Denote by $\mathbb{P}_{bin}$ and $\mathbb{F}_{bin}$ the binomial PDF and CDF. The expected payoff of a peripheral firm is

$$V(d) = \alpha^d \zeta \times \sum_{f=0}^{\left\lfloor \frac{d\theta}{\beta} \right\rfloor} \mathbb{P}_{bin}[f; d, 1-\zeta](d\theta - \beta f)$$

$$= d\alpha^d \zeta\theta \times \left\{ \mathbb{F}_{bin}\left[\left\lfloor \frac{d\theta}{\beta} \right\rfloor; d, 1-\zeta'\right] - \frac{\beta(1-\zeta')}{\theta} \mathbb{F}_{bin}\left[\left\lfloor \frac{d\theta}{\beta} \right\rfloor - 1; d-1, 1-\zeta'\right] \right\}. \quad (8)$$

This is a simpler functional form than equation (4). In equation (4), each counter-party carries a risk of default given by $1-\alpha^d$, which is endogenously determined by $d$. Here in equation (8), counter-parties are resilient core firms, so counter-party default risk is given by $1-\zeta'$. Unfortunately, there is no closed-form formula for $d^*$ even in this simpler case, and we provide simulations in Figure 12. Firms naturally form more links for larger $\alpha$ and $d^*$ increases. What is striking is that even under weak contagion (small $\beta$), even without any cliques (large $m$ and singletons within the periphery), even without any endogenous counter-party risk ($1-\zeta'$ term in equation (8) instead of the $1-\alpha^d$ term in equation (4)), the increase in the degree $d^*$ offsets the gain from risk $1-\alpha$, and systemic risk increases as $\alpha$ increases. Even mild network externalities generate the volatility paradox in the case of core-periphery as well.

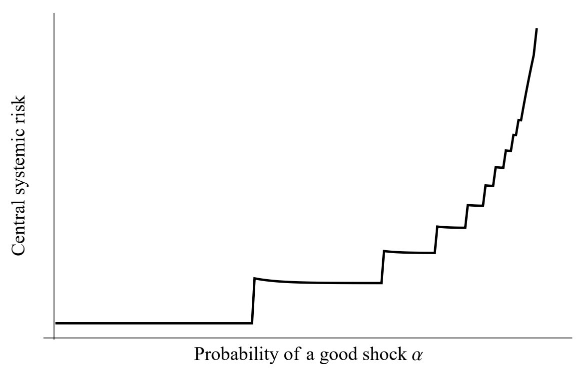



Figure 12: Central volatility paradox persists under weak contagion, $\theta = 1$, $\beta = 2$.

# 4 Core-periphery with ex-ante identical agents

As remarked upon in the introduction a number of explanations have been offered for the emergence of the core-periphery structure. Within the contagion context, Farboodi (2015) explains the emergence of this structure in the interbank lending market by positing that not all banks have investment opportunities. Erol (2017) argues that bailouts mitigate contagion, which allows some banks to become systemically important and leads to a core-periphery structure. Both papers rely on ex-ante heterogeneous agents. Here, we outline how a core-periphery structure with ex-ante identical agents emerges in our model due to lack of coordination across lenders.

We enlarge the setting described in subsections 2.1.2 and 3.1.2 by endogenizing the interest rates that firms face. We establish the existence of a group stable network in which some firms face low interest rates and the remaining firms face high interest rates despite that fact that all firms are identical. To avoid taxing the reader's memory, we restate the assumptions needed from the earlier subsections for the special case of ex-ante identical agents. All assumptions and results in this section will be flagged by the letter 'E'.

Formally, consider $n$ identical firms. Each firm has access to a unit mass of risk neutral lenders, each of whom has $L$ to lend to the firm. Denote by $R_i$ the interest rate that firm $i$ faces. The lenders are infinitesimal, so they take interest rates as given and decide independently of each other to lend or not. Simultaneously with lending between lenders and firms, firms form a network of joint bilateral investments among themselves. $P$ is the return per unit investment into the projects, $\overline{C} > \underline{C}$ are high and low management costs, $K$ is the high operational cost (low operational cost is 0), $\overline{B} > \underline{B}$ are high and low returns from projects, $\alpha$ is the probability of having low management cost for a project, and $\zeta$ is the probability of having a low operational cost. All these parameters are identical across all firms.

An *equilibrium* is defined as interest rates $R_i$ for each firm $i$ and a network among $n$ firms, such that the following two statements hold:

- Given the network formed, $R_i$ makes the lenders of firm $i$ indifferent between lending or not.
- Given interest rates $\{R_i\}_{i \leq n}$, the network formed is group stable.

**Assumption E 1.** *High operational costs are large enough to force firms into default regardless of the interest rate:*

$$K > L\left(\frac{\overline{B} - \underline{C}}{P} - 1\right)$$

**Assumption E 2.** *A firm with the lowest possible interest rate $\zeta^{-1}$ faces no default risk other than operational risk (node shock):*

$$\frac{\underline{B} - \overline{C}}{P} > \frac{1}{\zeta}$$

**Assumption E 3.** *There exists a high interest rate facilitating high default risk that is individually rational for lenders:*

$$\frac{\overline{B}-\underline{C}}{P} > \frac{1}{\zeta\left(\alpha\zeta\right)^{\left\lfloor \frac{1}{1-\alpha\zeta}\right\rfloor}} > \frac{\overline{B}-\underline{C}}{P} - \frac{min\left\{\overline{B}-\underline{B},\overline{C}-\underline{C}\right\}}{L}.$$

**Theorem E 1.** *Suppose that $n > \left\lfloor \frac{1}{1-\alpha\zeta}\right\rfloor$. Then, for any $m > \left\lfloor \frac{1}{1-\alpha\zeta}\right\rfloor$, there exists an equilibrium such that $m$ many firms face interest rate $R' = \zeta^{-1}$. The remaining firms face interest rate $R = \zeta^{-1}\left(\alpha\zeta\right)^{-\left\lfloor \frac{1}{1-\alpha\zeta}\right\rfloor}$. The network is core-periphery as described in Theorem 6, where the former $\left\lfloor \frac{1}{1-\alpha\zeta}\right\rfloor$ firms with interest rate $R'$ are the core and the rest with interest rate $R$ are the periphery.*

A high interest reduces the profit margins of a peripheral firm, making it unlikely to repay its debt. Then, an individual lender of a periphery firm charges a high premium because the firm has a high default probability. The heterogeneity in premiums arises in equilibrium due to lack of coordination across lenders. The heterogeneity in premiums, and therefore margins, renders some firms more resilient than others and the resilient forms occupy the core. The resulting network and the resulting default probabilities end up being consistent with the premiums charged. Therefore, a core-periphery structure emerges from a coordination failure across lenders, although all firms are ex-ante identical.

This highlights an inefficiency related to the core-periphery when prices are endogenized. Fundamentally, $\zeta^{-1}-1$ is the premium that a firm must inevitably pay. $\zeta\left(1-\left(\alpha\zeta\right)^{\left\lfloor \frac{1}{1-\alpha\zeta}\right\rfloor}\right)$ is the probability that a peripheral firm defaults solely due to contagion. Thus, an excess "contagion premium" $\zeta^{-1}\left(\left(\alpha\zeta\right)^{-\left\lfloor \frac{1}{1-\alpha\zeta}\right\rfloor}-1\right)$ must be paid to lenders. In fact, more firms could have enjoyed low premiums and the network formed would facilitate less contagion. In terms of Theorem E1, larger $m$ would render less firms exposed to contagion. On top of the inefficient contagion, the risk of contagion reduces the number of projects undertaken, which creates an opportunity cost as the second layer of inefficiency related to the core-periphery.

The network formed in Theorem E1 features a large core with regards to Theorem 6. Peripheral firms have links only with core firms. There are other equilibria in which fewer firms face low interest rates and the network formed has a small core with regards to Theorem 6. In such equilibria, the periphery is formed into disjoint cliques besides their links with the core. In this case, the interest rate faced by periphery firms does not have a closed-form solution, and so establishing this result requires introducing heavy notation and a modification to Assumption E3. We skip this for brevity.

# 5 Conclusion

This paper introduced a model of endogenous network formation and systemic risk which furnishes three lessons. First, the network formed critically depends on the correlation between shocks to links. Misconceiving this correlation significantly underestimates the probability of systemwide default. Second, even if the economy becomes fundamentally safer (the probability of good shocks increases), the probability of system-wide default can increase. Systemic risk increases because of the endogenous correlation of default risk that stems from network formation. Finally, in spite of the fact that the core-periphery structure facilitates systemic risk because the core with its dense interconnections encourages contagion, they arise in spite of it.

The formation results are of independent interest as they apply to any setting where an agents' payoffs are an arbitrary function of its degree in a network but are strictly decreasing in the number of links in the connected component she is a member of. Ours is one example of such a setting, but it is easy to imagine others.

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# Appendix A Proofs

*Proof.* **(Proposition 1)** By Assumption 2, $d_i\theta \leq \tilde{d}\theta < \gamma$. Therefore, one bad shock forces a firm into default. Also $d_i\theta \leq \tilde{d}\theta < \beta$, so that one defaulting counter-party forces a firm into default. That is, either all shocks are good in the component and all firms continue their business, or at least one shock is bad and all firms default. There are $e$ links in the component. For each link, two shocks, one for each incident firm, realizes. So $2e$ shocks realize in the component. There is an $\alpha^{2e}$ probability that the component survives in which case the firm with degree $d$ has payoff $d\theta$. Otherwise, the firm defaults and earns 0. Thus, the expected payoff of a firm is $d\alpha^{2e}\theta$. □

*Proof.* **(Proposition M 1)** *Notation:* For all $i$ and for all $j \in D_i$, set $a_{ij} = 1$ if firm $i$ keeps project $\{i, j\}$, and $a_{ij} = 0$ otherwise. Denote by $a_i = (a_{ij})_{j \in D_i}$ the strategy of $i$. Let $a = (a_i)_{i \leq n}$ be the action profile played by all firms. Denote by $a_{-i} = (a_j)_{j \neq i}$ the action profile played by firms other than $i$. Denote by $\mathbf{1}$ the vector of 1's and $\mathbf{0}$ the vector of 0's.

*Ex-post payoffs:* For $X \in \{0, 1\}$ and $Y \in \left\{\overline{C}, \underline{C}\right\}$, let

$$D^i_{X,Y}(a) = \left|\left\{ j \in D_i \middle| a_{ji} = X, C^{ij} = Y \right\}\right|.$$

This is the number of counter-parties $j$ of $i$ such that the counter-party $j$ is playing $X$ for its project $\{i, j\}$ with $i$, whereas $i$'s management cost for the project $\{i, j\}$ with $j$ is $Y$. Then

$$\left(\left|D^i_{1,\underline{C}}\right| + \left|D^i_{1,\overline{C}}\right|\right)\overline{B} + \left(\left|D^i_{0,\underline{C}}\right| + \left|D^i_{0,\overline{C}}\right|\right)\underline{B}$$

is the revenue of firm $i$. Also, $d_iPR$ must be repaid back to lenders. Let

$$\Pi_i(a) = \left(\left|D^i_{1,\underline{C}}\right| + \left|D^i_{1,\overline{C}}\right|\right)\overline{B} + \left(\left|D^i_{0,\underline{C}}\right| + \left|D^i_{0,\overline{C}}\right|\right)\underline{B} - d_iPR.$$

Since firms are protected by limited liability, $i$ has profit $(\Pi_i(a))^+$. On top of the profit, $i$ incurs, the effort cost of managing projects (operational cost), is

$$C_i(a) = \left(\left|D^i_{1,\overline{C}}(a)\right| + \left|D^i_{0,\overline{C}}(a)\right|\right)\overline{C} + \left(\left|D^i_{0,\underline{C}}\right| + \left|D^i_{1,\underline{C}}\right|\right)\underline{C}.$$

Accordingly, $i$'s payoff is

$$U_i(a) = (\Pi_i(a))^+ - C_i(a).$$

$U_i(a)$ can fall below zero because the management costs of projects, $\underline{C}$ and $\overline{C}$, are utility costs of effort. If $i$ continues with projects despite having very small or zero profit, $U_i$ can be negative.

*Best responses:* Define

$$V_i(a) = \Pi_i(a) - C_i(a).$$

By Assumption M 1, for any $a_{-i}$, $V_i(a_i', a_{-i})$ is increasing in $a_i'$. In particular, for any $a_{-i}$, $\arg\max_{a_i'} V_i(a_i', a_{-i}) = \mathbf{1}$ is the unique maximizer.

Case 1: If $V_i(\mathbf{1}, a_{-i}) > 0$, then $\Pi_i(\mathbf{1}, a_{-i}) > C_i(\mathbf{1}, a_{-i}) > 0$. So $U_i(\mathbf{1}, a_{-i}) = V_i(\mathbf{1}, a_{-i}) > 0$. Consider any $a_i' < \mathbf{1}$. If $\Pi_i(a) \leq 0$, then $U_i(a_i', a_{-i}) \leq 0 < U_i(\mathbf{1}, a_{-i})$. If $\Pi_i(a) > 0$, then $U_i(a_i', a_{-i}) = V_i(a_i', a_{-i}) < V_i(\mathbf{1}, a_{-i}) = U_i(\mathbf{1}, a_{-i})$. Thus $a_i' = \mathbf{1}$ is the unique best response.

Case 2: If $V_i(\mathbf{1}, a_{-i}) < 0$, then $V_i(a_i', a_{-i}) < 0$ for all $a_i'$. Then $\Pi_i(a_i', a_{-i}) < C_i(a_i', a_{-i})$ for all $a_i'$. Also, for all $a_i' \neq \mathbf{0}$, $0 < C_i(a_i', a_{-i})$. Then combining both, for all $a_i' \neq \mathbf{0}$, $(\Pi_i(a_i', a_{-i}))^+ < C_i(a_i', a_{-i})$. So $U_i(a_i', a_{-i}) < 0$ for all $a_i' \neq \mathbf{0}$. Then the unique best response is $a_i' = \mathbf{0}$ since it yields $U_i(\mathbf{0}, a_{-i}) = 0$.

Case 3: If $V_i(\mathbf{1}, a_{-i}) = 0$, then $V_i(a_i', a_{-i}) < 0$ for all $a_i' \neq \mathbf{1}$. Similar to Case 2, we get $U_i(a_i', a_{-i}) < 0$ for all $a_i' \neq \mathbf{0}, \mathbf{1}$. For $a_i' = \mathbf{0}$, $U_i(\mathbf{0}, a_{-i}) = 0$. For $a_i' = \mathbf{1}$, by $V_i(\mathbf{1}, a_{-i}) = 0$, we have $\Pi_i(\mathbf{1}, a_{-i}) = C_i(\mathbf{1}, a_{-i}) > 0$. So $U_i(\mathbf{1}, a_{-i}) = V_i(\mathbf{1}, a_{-i}) = 0$. Then both $a_i' \in \{\mathbf{0}, \mathbf{1}\}$ are the best responses.

This establishes that the best response is either to keep all projects ($a_i' = \mathbf{1}$ when $V_i(\mathbf{1}, a_{-i}) \geq 0$) or to drop all projects ($a_i' = \mathbf{0}$ when $V_i(\mathbf{1}, a_{-i}) \leq 0$). Observe that

$$\begin{aligned} V_i(\mathbf{1}, a_{-i}) = {} & \left(\left|D^i_{1,\underline{C}}(\mathbf{1}, a_{-i})\right| + \left|D^i_{1,\overline{C}}(\mathbf{1}, a_{-i})\right|\right)\overline{B} + \left(\left|D^i_{0,\underline{C}}(\mathbf{1}, a_{-i})\right| + \left|D^i_{0,\overline{C}}(\mathbf{1}, a_{-i})\right|\right)\underline{B} - d_i PR \\ & - \left(\left(\left|D^i_{1,\overline{C}}(\mathbf{1}, a_{-i})\right| + \left|D^i_{0,\overline{C}}(\mathbf{1}, a_{-i})\right|\right)\overline{C} + \left(\left|D^i_{0,\underline{C}}(\mathbf{1}, a_{-i})\right| + \left|D^i_{1,\underline{C}}(\mathbf{1}, a_{-i})\right|\right)\underline{C}\right) \\ = {} & d\left(\overline{B} - \underline{C} - P\right) - \left(\left|D^i_{0,\overline{C}}(\mathbf{1}, a_{-i})\right| + \left|D^i_{0,\underline{C}}(\mathbf{1}, a_{-i})\right|\right)\left(\overline{B} - \underline{B}\right) \\ & - \left(\left|D^i_{1,\underline{C}}(\mathbf{1}, a_{-i})\right| + \left|D^i_{0,\underline{C}}(\mathbf{1}, a_{-i})\right|\right)\left(\overline{C} - \underline{C}\right) \\ = {} & d_i\theta - f_i\beta - b_i\gamma. \end{aligned}$$

□

*Proof.* **(Proposition 2)** Corollary of Proposition 1 in Erol and Vohra (2020) □

*Proof.* **(Theorem 1)** Fix a network that consists of disjoint cliques. We consider all types of deviations by pairs from this network.

*Case 1: Unilateral deviation.* Consider firm $i$ and a deviation in which $i$ cuts $d_i - y$ links and now has $y$ links. There are $y + \frac{(d_i-1)d_i}{2}$ links left in the component. Then, $i$'s payoff becomes

$y\alpha^{2y+(d_i-1)d_i}$. This is a profitable deviation if and only if

$$y\alpha^{2y+(d_i-1)d_i} > d_i\alpha^{(d_i-1)d_i} \iff y\alpha^{2y} > d_i\alpha^{2d_i}.$$

Observe that the function $F(x) = x\alpha^{2x}$ is log-concave, and so it is single-peaked. Single-peakedness and $y < d_i$ implies that

$$y\alpha^{2y} > d_i\alpha^{2d_i} \implies (d_i - 1)\,\alpha^{2(d_i-1)} > d_i\alpha^{2d_i} \iff d_i > \left(1-\alpha^2\right)^{-1}.$$

$y$ could be equal to $d_i - 1$. Therefore, there does not exist any unilateral deviation (i.e., any $y$) if and only if $d_i \leq (1-\alpha^2)^{-1}$ for all cliques.

*Case 2 : Deviation by a pair in the same clique.* Consider $i$ and $j$ in the same clique (so $d_i = d_j$) and a deviation in which $i$ cuts $d_i - y$ links and now has $y$ links, whereas $j$ cuts $d_i - z$ links and now has $z$ links.

We can assume $i$ and $j$ have not cut their links $\{i, j\}$: instead of cutting the link $\{i, j\}$ and some more links with other firms in the clique, $i$ and $j$ could keep the link and cut one more link with other firms in the clique. Hence, there exists a profitable deviation by $i$ and $j$ if and only if there exists a profitable deviation in which $i$ and $j$ keep their own link. So we can suppose that $i$ and $j$ keep their link and there are $y + z - 1 + \frac{(d_i-2)(d_i-1)}{2}$ links left in the component.

Then, $i$'s payoff becomes $y\alpha^{2y+2z-2+(d_i-2)(d_i-1)}$ while $j$'s payoff becomes $z\alpha^{2y+2z-2+(d_i-2)(d_i-1)}$. Without loss of generality, suppose that $y \leq z$. This is a profitable deviation if and only if

$$y\alpha^{2y+2z-2+(d_i-2)(d_i-1)} > d_i\alpha^{(d_i-1)d_i} \wedge z\alpha^{2y+2z-2+(d_i-2)(d_i-1)} > d_i\alpha^{(d_i-1)d_i} \iff$$

$$y\alpha^{2y+2z-2+(d_i-2)(d_i-1)} > d_i\alpha^{(d_i-1)d_i} \iff y\alpha^{2y+2z} > d_i\alpha^{4d_i}.$$

If $z = y$ there exists a profitable deviation in case 2 if and only if there exists $y$ such that $y\alpha^{4y} > d_i\alpha^{4d_l}$. As in case 1, single-peakedness and $y < d_i$ implies that

$$y\alpha^{4y} > d_i\alpha^{4d_i} \implies (d_i - 1)\,\alpha^{4(d_i-1)} > d_i\alpha^{4d_i} \iff d_i > \left(1-\alpha^4\right)^{-1}.$$

Again, as in case 1, $y$ could be equal to $d_i - 1$. In this case there does not exist any profitable deviation by a pair of firms in the same clique if and only if $d_i \leq (1-\alpha^4)^{-1}$ for all cliques.

*Case 3: Deviation by a pair in different cliques.* Consider $i$ and $j$ in different cliques who deviate jointly. If $i$ and $j$ do not form a link, then, this deviation must consist of two unilateral deviations in which both $i$ and $j$ increase their individual utilities by cutting some of their links in their own cliques. This is considered in case 1. So we can assume that case 3 consists of deviations in which the deviators $i$ and $j$ do form a link.

Consider the deviation in which $i$ cuts $d_i - y$ links and adds the link with $j$. This leaves $i$ with $y+1$ links. After $j$ cuts $d_i - z$ links and adds the link with $i$ it will have $z+1$ links. After the deviation, in the component containing $i$ and $j$, there are $2y + (d_i - 1)\, d_i + 2 + 2z + (d_j - 1)\, d_j$ links.

We first show that if $i$ or $j$ fail to preserve all of their existing links, i.e. $y \neq d_i$ or $z \neq d_j$, this will not be a profitable deviation. Without loss of generality, suppose that $y \neq d_i$. Then, $y + 1 \leq d_i$. Therefore, $i$'s payoff must be strictly larger than its previous payoff for this to be a profitable deviation:

$$(y+1)\, \alpha^{2y+(d_i-1)d_i+2+2z+(d_j-1)d_j} > d_i \alpha^{d_i(d_i+1)}. \tag{9}$$

From case 1 we know it suffices to consider networks in which all degrees are less than $\left(1-\alpha^2\right)^{-1}$. That is, we can assume $d_i \leq \left(1-\alpha^2\right)^{-1}$. Then $y+1 \leq d_i \leq \left(1-\alpha^2\right)^{-1}$. By single-peakedness of $F(x) = x\alpha 2x$, we have $(y+1)\, \alpha^{2y+2} \leq d_i \alpha^{2d_i}$. Substituting this back into Equation (9) we get

$$d_i \alpha^{2d_i} \alpha^{(d_i-1)d_i+2z+(d_j-1)d_j} \geq (y+1)\, \alpha^{2y+(d_i-1)d_i+2+2z+(d_j-1)d_j} > d_i \alpha^{d_i(d_i+1)}$$

$$\implies \alpha^{2z+(d_j-1)d_j} > 1.$$

which is a contradiction.

Accordingly, we can assume that case 3 consists of deviations in which the deviating pair $i$ and $j$ maintain all their links in their own cliques, i.e. $y = d_i$ and $z = d_j$, and add the missing link $\{i, j\}$ between them. This is a profitable deviation if and only if

$$(d_i+1)\, \alpha^{d_i(d_i+1)+2+d_j(d_j+1)} > d_i \alpha^{d_i(d_i+1)} \wedge (d_j+1)\, \alpha^{d_j(d_j+1)+2+d_i(d_i+1)} > d_j \alpha^{d_j(d_j+1)}$$

$$\iff (d_i+1)\, \alpha^{2+d_j(d_j+1)} > d_i \wedge (d_j+1)\, \alpha^{2+d_i(d_i+1)} > d_j$$

$$\iff d_i < \left(1-\alpha^{2+d_j(d_j+1)}\right)^{-1} - 1 \wedge d_j < \left(1-\alpha^{2+d_i(d_i+1)}\right)^{-1} - 1.$$

Define $\psi : [0, \infty) \to \mathbb{R}$ as $\psi(x) := \left(1-\alpha^{2+x(x+1)}\right)^{-1} - 1$. Then, there is a profitable deviation we need to consider in case 3 if and only if there are two separate cliques, with order $d_i + 1$ and $d_j + 1$ such that

$$d_i < \psi(d_j) \wedge d_j < \psi(d_i).$$

Note that $\psi$ is strictly decreasing, $\psi(0) = (1-\alpha^2)^{-1} - 1 > 0$ and $lim_{x \to +\infty} = 0$. Denote by $\underline{d}$ the unique fixed point of $\psi$: $\psi(\underline{d}) = \underline{d}$. Observe that $\psi^{-1}$ is well defined on the interval $(0, \psi(0)]$. Define $\phi : [0, \psi(0)] \to \mathbb{R}$ as

$$\phi(x) = \begin{cases} \psi(0) & if\, x = 0, \\ min\left\{\psi(x), \psi^{-1}(x)\right\} & if\, x \in (0, \psi(0)], \end{cases}$$

$\phi$ is strictly decreasing and $\phi$'s unique fixed point is $\underline{d}$. Let $\overline{d} = (1-\alpha^4)^{-1}$. Next, we complete the argument that verifies the necessary and sufficient conditions given in the Theorem.

For necessity suppose first that $d_i < \underline{d}$ and $d_j < \underline{d}$. Since $\psi$ is strictly decreasing, it follows that $\psi(d_i) > \psi(\underline{d}) = \underline{d} > d_j$ and similarly $\psi(d_j) > d_i$. Then, $i$ and $j$ would deviate. Second, suppose the clique that contains $i$ is the smallest in the network ($d_i = \underline{d}_G$ ). Suppose that $\underline{d}_G < \underline{d}$ and $d_j < \phi(\underline{d}_G)$. Then

$$d_j < \phi(\underline{d}_G) = min\left\{\psi(\underline{d}_G), \psi^{-1}(\underline{d}_G)\right\} \implies$$

$$d_j < \psi(\underline{d}_G) \wedge d_j < \psi^{-1}(\underline{d}_G) \implies$$

$$d_j < \psi(\underline{d}_G) \wedge \psi(d_j) > \underline{d}_G.$$

Then $i$ and $j$ have a profitable deviation. In sum, conditional on all degrees being less than $\overline{d}$, if there is no profitable deviation, there are no two cliques with orders strictly less than $\underline{d}+1$ or there is exactly one order strictly less than $\underline{d}+1$ but all others have orders larger than $\phi(\underline{d}_G)+1$. Now we show that, conditional on all degrees being less than $\overline{d}$, this condition is sufficient as well.

Suppose that $d_i \geq \underline{d}$ and $d_j \geq \underline{d}$. Then $d_i \geq \underline{d} \implies \psi(d_i) \leq \psi(\underline{d}) = \underline{d} \leq d_j$. Accordingly, there are no profitable deviations between cliques that have size larger than $\underline{d}+1$. Therefore, if all cliques have size larger than $\underline{d}+1$, there are no profitable deviations. Suppose that all but one have size larger than $\underline{d}+1$. Let $d_i+1$ be the size of the remaining clique, which has to be smallest clique: $d_i = \underline{d}_G$. Suppose that $d_j \geq \phi\left(\overline{d}_G\right)$ for all other $d_j$'s. Then

$$d_j \geq \phi(\underline{d}_G) = min\left\{\psi(\underline{d}_G), \psi^{-1}(\underline{d}_G)\right\} \implies$$

$$d_j \geq \psi(\underline{d}_G) \vee d_j \geq \psi^{-1}(\underline{d}_G) \implies$$

$$d_j \geq \psi(\underline{d}_G) \vee \psi(d_j) \geq \underline{d}_G.$$

Thus, there are no profitable deviations that involve the smallest clique. All other cliques have size larger than $\underline{d}+1$, so they also have no profitable deviations among each other.

Hence, we have shown in case 3 there are no profitable deviations (that have not been accounted for in other cases) if and only if either all degrees are larger than $\underline{d}$, or one clique has order less than $\underline{d}+1$, say but all the others have it larger than $\phi\left(\underline{d}_G\right)+1$. Combining all three cases concludes the proof for the necessary and sufficient condition.

For existence, L'Hopital's rule can be used to show that the gap $\overline{d}-\underline{d}$ grows unboundedly as $\alpha$ grows. At some point, $\overline{d}-\underline{d}\geq 2$ and so there must exist two integers in the interval $\left[\underline{d},\overline{d}\right]$. Take two of these integers, $\lceil\underline{d}\rceil$ and $\lceil\underline{d}\rceil+1$. If $n\geq\left(\lceil\underline{d}\rceil+1\right)^2$ then

$$n-\left\lfloor\frac{n}{\lceil\underline{d}\rceil+1}\right\rfloor\left(\lceil\underline{d}\rceil+1\right)\leq\lceil\underline{d}\rceil<\lceil\underline{d}\rceil+1\leq\left\lfloor\frac{n}{\lceil\underline{d}\rceil+1}\right\rfloor\implies$$

$$\left\lfloor\frac{n}{\lceil\underline{d}\rceil+1}\right\rfloor\left(\lceil\underline{d}\rceil+2\right)-n>0.$$

Then, the network consists of

- $n-\left\lfloor\frac{n}{\lceil\underline{d}\rceil+1}\right\rfloor\left(\lceil\underline{d}\rceil+1\right)$ many cliques of order $\lceil\underline{d}\rceil+2$ and
- $\left\lfloor\frac{n}{\lceil\underline{d}\rceil+1}\right\rfloor\left(\lceil\underline{d}\rceil+2\right)-n$ many cliques of order $\lceil\underline{d}\rceil+1$

is well-defined and bilaterally stable. □

*Proof.* **(Proposition 3)** By Proposition 2 a group stable network (if it exists) is composed of disjoint cliques. The payoff to a firm in a $(d+1)$-clique is $V(d)$. First, no clique can have order $d+1>d^*+1$ in the realized network. Otherwise, $d^*+1$ members would deviate by forming a $(d^*+1)$-clique and cutting all other links. This would be a strict improvement since $d^*$ is the unique maximizer of $V(d)$. Second, there can be at most one clique which has order strictly less than $d^*+1$. Recall that $V\left(d\right)$ is single-peaked, and so it is increasing up to $d^*$ over integers. Then, members of separate cliques each of which has less than $d^*$ order would deviate to forming a larger clique (up to order $d^*+1$) to get their degree closer to $d^*$ and improve their payoffs. These two observations imply that there must be as many $(d^*+1)$-cliques as possible in the network, and the remaining part must also be a clique. □

*Proof.* **(Theorem 2)** Corollary of Theorem 1 in Erol and Vohra (2020) □

*Proof.* **(Theorem 3)** Fix an efficient network $G$ and select from it a component with $e$ links and call it $S$. The total payoff of firms in the component $S$ is $2e\alpha^{2e}$. Define $W$ :

$\mathbb{R}^+ \to \mathbb{R}^+$ as $W(x) = 2x\alpha^{2x}$. Note that $W$ is log-concave, and so it is single-peaked. Set $k^* = argmax_{x\in\mathbb{N}} W(x)$ as the maximizer of $W$ over integers. Since $W$ is single-peaked, if $e > k^*$, by deleting one link within $S$, the total payoff increases. So $G$ cannot be efficient. Hence, $e$ must satisfy $e \leq k^*$. If $e < k^*$ and if there is an absent link within component $S$, then, by adding that link, the total payoff increases. So $G$ cannot be efficient. Therefore, either $e = k^*$ or there are no absent links in the component $S$, i.e. $S$ is a clique.

Suppose that $S$ is not a clique. Then, we must have $e = k^*$. Let $r$ be the largest integer such that $(0.5)r(r-1) + 1 \leq k^* \leq (0.5)r(r+1)$. Note, there at least $r+1$ firms in $S$. Otherwise, there cannot be more than $(0.5)r(r-1)$ links in $S$. Denote the average of payoffs of firms in $S$ by $u$. Since there are at least $r+1$ firms in $S$ and the sum of payoffs in $S$ is $2k^*\alpha^{2k^*}$, we have $u \leq \frac{2k^*\alpha^{2k*}}{r+1}$.

*Case 1:* $k^* \leq (0.5)\left(r^2 - 1\right)$. Then, $u = \frac{2k^*\alpha^{2k*}}{r+1} \leq (r-1)\alpha^{2k^*}$. By definition of $r$, $(0.5)r(r-1) < k^*$, and so $u < (r-1)\alpha^{r(r-1)} = V(r-1) \leq V(d^*)$.

*Case 2:* $(0.5)r^2 \leq k^* \leq (0.5)r(r+1) - 1$. $k^*+1$ is an integer as well as $(0.5)r(r+1)$. Recall that $k^*$ is the maximizer of $W$ over integers. And so we have $W(k^*) > W(k^*+1)$. Since $V$ is single-peaked on real numbers, $W(k^*) > W(k^*+1)$ implies that $W(x)$ is decreasing for real numbers $x \in [k^*+1, \infty]$. In particular, $W((0.5)r(r+1)) \geq W((0.5)r(r+2))$. Then we have

$$W((0.5)r(r+1)) = r(r+1)\alpha^{r(r+1)} > W((0.5)r(r+2)) = r(r+2)\alpha^{r(r+2)}$$

$$\implies r+1 > (r+2)\alpha^{r} \implies (r+1)\alpha^{(r-1)r} > (r+2)\alpha^{r^2}$$

$$\implies V(r-1) = (r-1)\alpha^{(r-1)r} > \frac{(r-1)(r+2)}{(r+1)}\alpha^{r^2}. \tag{10}$$

$k^* \leq (0.5)r(r+1) - 1$ so $u = \frac{2k^*\alpha^{2k*}}{r+1} \leq \frac{r(r+1)-2}{r+1}\alpha^{2k^*}$. $(0.5)r^2 \leq k^*$ so $\alpha^{2k^*} \leq \alpha^{r^2}$. Thus, $u \leq \frac{r(r+1)-2}{r+1}\alpha^{r^2} = \frac{(r+2)(r-1)}{r+1}\alpha^{r^2}$. By Equation (10) we have $u < V(r-1) \leq V(d^*)$.

*Case 3:* $k^* = (0.5)r(r+1)$. Now $S$ is a component that has $(0.5)r(r+1)$ links but it is not a clique, so, there must be at least $r+2$ firms in $S$. Then, $u \leq \frac{2k^*\alpha^{2k*}}{r+2} < \frac{2k^*\alpha^{2k*}}{r+1} = r\alpha^{r(r+1)} = V(r) \leq V(d^*)$.

By combining the three cases, we see that if $S$ is not a clique, the average payoff in $S$ is strictly less than $V(d^*)$. But a group stable network achieves an average payoff of $V(d^*)$. So, in any efficient network, the average payoff must be exactly $V(d^*)$ and each component is a cliques. In networks that consist of disjoint cliques, firm $i$ has payoff $V(d_i)$, which is strictly less than $V(d^*)$ if $d_i \neq d^*$. Therefore, the efficient network consists of disjoint cliques and all firms have degree $d^*$. That is, the unique efficient network is the the unique group stable network. □

*Proof.* **(Proposition 4)** In any given network, if all shocks are bad, then, all firms default and if all shocks are good, then, all firms continue. The payoff of a firm with $d$ links is $d\theta$ or 0 respectively. Thus, the expected payoff of each firm is $d\sigma\theta$. Then, it is clear that in a group stable (or pairwise stable) network there cannot be any missing links because that would lead to a profitable pairwise deviation. The only candidate is the complete network which is clearly group stable. □

*Proof.* **(Theorem 4)** Under popcorn, for any $\sigma \in (0,1)$, $\left(\tilde{d}+1\right)$-cliques are formed. So, the systemic risk, $1-\sigma$, can take any value between 0 and 1. Under dominoes, the necessary and sufficient condition for a complete network to be formed is, is $V\left(\tilde{d}\right) \geq V\left(\tilde{d}-1\right)$. This is because $V$ is single-peaked. So, a complete network is formed if and only if

$$V\left(\tilde{d}\right) = \tilde{d}\alpha^{\tilde{d}\left(\tilde{d}+1\right)} \geq V\left(\tilde{d}-1\right) = \left(\tilde{d}-1\right)\alpha^{\left(\tilde{d}-1\right)\tilde{d}}$$

$$\iff \alpha \geq \left(\frac{\tilde{d}-1}{\tilde{d}}\right)^{\frac{1}{2\tilde{d}}} \iff \left(1-\left(\frac{\tilde{d}-1}{\tilde{d}}\right)^{\frac{\tilde{d}+1}{2}}\right)^{\frac{n}{\tilde{d}+1}} \geq \left(1-\alpha^{\tilde{d}\left(\tilde{d}+1\right)}\right)^{\frac{n}{\tilde{d}+1}}.$$

□

*Proof.* **(Proposition M 2)** Assumption M 3 ensures that a firm with a bad operational shock that continues operations cannot make positive payoffs. So, such a firm terminates operations and drops all projects in order to save on the project management costs. The rest of the proof is identical to that of Proposition 1. □

*Proof.* **(Proposition 5)** As before, a normal firm $i$ with a good operational shock $i$ defaults if any of its links are bad, or any of its counterparties default. This argument can be iterated along any normal path. Therefore, there are two ways that $i$ can default. One is that any firm that can be reached from $i$ by a normal path gets a bad operational shock, and so all normal firms along the normal path default sequentially. Second, a link of a normal firm $j$ that can be reached from $i$ via a normal path is bad for $j$, which makes $j$ default, and so all normal firms along the normal path from $j$ to $i$ sequentially default. The first type of contagion is not triggered by a resilient firm with probability $(\zeta')^{d_i^M}$ and it does not get triggered by a normal firm with probability $\zeta^{d_i^N}$. The second type of contagion is not triggered with probability $\alpha^{2e_i^N+e_i^M}$. Here $e_i^M$ is not multiplied by 2 because one firm incident to this link is a resilient firm that does not default due to link shocks. Therefore, the probability that a normal $i$ does not default is

$$p_i = \zeta^{d_i^N}\left(\zeta'\right)^{d_i^M}\alpha^{2e_i^N+e_i^M}$$

and the expected payoff of $i$ is $p_i d_i \theta$.

As for a resilient firm $i$, if $i$ has low operational costs, $i$ continues with all projects. This is because $\theta' > \beta' + \gamma'$ which is implied by Assumption M 4. Therefore, correlations of the default probabilities of $i$'s counterparties do not factor into $i$'s payoff. The only risk that $i$ defaults is that it has high operational costs. Accordingly, $i$'s conditional expected payoff is the sum of its expected payoffs from each of its links. All of $i$'s resilient counterparties with low operational shocks continue. Accordingly, conditional on $i$ having low operational shocks, $i$ has $\theta' - \alpha'\gamma' - \zeta'\beta'$ expected payoff for each of its resilient counterparties. $\alpha'\gamma'$ is the expected loss due to the fact that the link can be bad for $i$. $\zeta'\beta'$ is the expected loss due to the fact that its resilient counterparty can have high operational cost. As for a normal counterparty of $i$, say $j \in N \cap D_i$, the probability that $j$ defaults conditional on $j$ having low operational cost is $p_j/\zeta'$. Accordingly, conditional on $i$ having low operational shocks, the expected loss of $i$ due to $j$'s default is $p_j\beta'/\zeta'$. Summing the expected payoffs and multiplying them by the probability $\zeta'$ that $i$ has low operational costs gives the expected payoff. □

*Proof.* **(Theorem 5)** Fix a network. A resilient firm $i$ always wants to link with all other firms. Accordingly, in any bilaterally stable network, all pairs of resilient firms have links. Suppose that there exists a normal firm $j$ that has a link with another normal firm $k$, but not with $i$. Then $j$ would sever its link with $k$, and $i$ and $j$ would form a link. This strictly improves $i$'s payoff. As for $j$, $p_j$ is reduced by $\alpha^2\zeta$ by cutting the link with $k$ and increased by $\zeta'\alpha$ by adding the link with $i$. Since $\zeta' > \zeta\alpha$, $p_j$ decreases strictly. $j$'s degree does not change, and so $j$'s payoff strictly improves. Therefore, in any bilaterally stable network, any normal firm with a normal counterparty must be counterparties with all resilient firms. So normal firms either have only resilient counterparties, or have resilient firms as counterparties and potentially other normal counterparties.

Take a normal firm $j$ that has a resilient counterparty $i$ and a normal counterparty $k$. If $k$ is not counterparties with $i$, $k$ would sever its link with $j$ and form a link with $i$. This would improve the payoffs of both $i$ and $k$. So $i$ and $k$ must already be counterparties. That is, all normal counterparties of $i$ must be counterparties with all resilient firms.

Now focus on the periphery. The proof of Proposition 2 applies, and so the periphery must be in cliques. Combine this with previous argument to see that a normal $i$ firm is either in a clique in the periphery where the entire clique is adjacent to all of the resilient firms, or $i$ does not have any normal counterparties and all of its counterparties are resilient firms. It turns out that these two cannot happen at the same time.

Consider the case of $m < d_M^*$. Take a normal firm $i$. Since $d\left(\zeta'\alpha\right)^d$ is single-peaked (due to log-concavity) and $m < d_M^*$, $i$ would want to form links with more resilient firms, and resilient firms would be happy to correspond. So in any bilaterally stable network, all normal firms must be counterparties of with all resilient firms. The periphery must be in disjoint cliques among themselves.

Consider the case of $m > d_M^*$. Normal firms would like to have $d_M^*$ many counterparties and there is sufficiently many resilient firms to form links with. Then all normal firms have $d_M^*$ many resilient counterparties and no normal counterparties. □

*Proof.* **(Theorem 6)** Consider the case $m < d_M^*$. In a group stable network, for a normal firm $i$, we have $d_i^M = m$ since all resilient firms can be immediately reached from $i$. Moreover, all normal firms $j$ that can be reached from $i$ via normal firms are also counterparties with all resilient firms, so that $e_i^M = md_i^N$. Therefore, $p_i = \zeta^{d_i^N} (\zeta')^m \alpha^{2e_i^N + md_i^N} = (\alpha^m \zeta)^{d_i^N} \alpha^{2e_i^N} \times (\zeta')^m$. Accordingly, $i$'s expected payoff is

$$d_i (\alpha^m \zeta)^{d_i^N} \alpha^{2e_i^N} (\zeta')^m \theta.$$

Note that $e_i^N$ is the number of links and $d_i^N$ is the number of firms in the component of $i$ in the subnetwork of normal firms. Accordingly, one can ignore all resilient firms, focus on only normal firms, and study group stable networks with the payoff function

$$U(d, e) = (d + m) (\alpha^m \zeta)^{d+1} \alpha^{2e}.$$

This payoff function is decreasing in $e$ and so the proof of Theorem 2 can be replicated to show the normal firms must be formed into disjoint cliques of order $d_N^* + 1$ and be connected to all resilient firms, and that there is no profitable deviation from this network.

Consider the case $m > d_M^*$. Now any bilaterally stable network is group stable. Normal firms have the maximum payoff they can achieve in any given network because they are connected to the least risky counterparties at their individual rational level. Normal firms can not be convinced to deviate. Clearly, the resilient firms do not want to alter their links among themselves. □

*Proof.* **(Theorem 7)** For $m < d_M^*$ it is clear that it is efficient that resilient firms are counterparties of all firms. For the rest, the proof of Theorem 3 can be replicated after modifying it to account for node shocks. Node shocks make no difference in the proof of Theorem 3.

For $m > d_M^*$, group stable network may not be efficient. Because it may be better to form more links between normal firms and resilient firms than what is individually rational for the normal firms. The loss in normal firms' payoffs can be compensated with the payoffs of resilient firms.

Notice that the gain that a resilient firm has by a link with a certainly defaulting normal firm, is $\theta' - \beta' - \gamma' > 0$. As we increase $m$, the potential gains from connecting all of the $n$ normal firms to all of the $m$ resilient firms increases at the order of $nm\left(\theta' - \beta' - \gamma'\right)$, which is unbounded.

If one wishes to bound from above the probability of default for a normal firm, one must bound the number of the normal firm's resilient counterparties because each resilient counterparty has operational risk. In the limit $m \to \infty$ for bounded $n$, putting an upper bound on the number of such links requires forgoing infinite total payoff to resilient firms for the sake of the bounded gains from a normal firm. That is, it is not efficient to limit the risk of normal firms. Then, given that the risk of normal firms must be is unlimited in the efficient network as $m \to \infty$, all normal firms face certainty of default in the limit, which means that all firms should be connected to all resilient firms to maximize the links between core and periphery to get $\theta' - \beta' - \gamma'$ from each links.

As for links between normal firms, there is a small but positive gain from not forming any links between normal firms. Accordingly, there is a bound $\underline{m}$ such that $m > \underline{m}$ requires efficiency to connect all firms to all resilient firms, and not form any links between normal firms. $\square$

*Proof.* **(Theorem E 3)** Note that Assumption E2 implies Assumption M 4, which implies Assumption M 1. Assumption E3 implies Assumption M 2. Assumption E1 and Assumption M 2 imply Assumption M 3. Then by Theorem6, the network formed is group stable.

In particular, $d_M^* = argmax\, d\left(\alpha\zeta\right)^d = \left\lfloor \frac{1}{1-\alpha\zeta} \right\rfloor < m$. Since $d_M^* < m$, $d_N^* = 0$. (The network is given by the case of the large core in Theorem6.) Then the default probability of a firm in the periphery is $1 - \zeta\left(\alpha\zeta\right)^{\left\lfloor \frac{1}{1-\alpha\zeta} \right\rfloor}$. The default probability of a firm in the core is $1-\zeta$. Then the interest rate that makes lenders of periphery firms indifferent is $\zeta^{-1}\left(\alpha\zeta\right)^{-\left\lfloor \frac{1}{1-\alpha\zeta} \right\rfloor} = R$. The interest rate that makes lenders of the core firms indifferent is $\zeta^{-1} = R'$. $\square$

# (MATERIAL FOR ONLINE APPENDIX)

## Appendix B Other measures of systemic risk

As we have mentioned in Section 2.3.1, our insights are valid under other measures of systemic risk as well. Here we formalize this. Given $\alpha$, the number of defaults is equal to $(d^* + 1)s$ with probability

$$\binom{\frac{n}{d^*+1}}{s} \left(1 - \alpha^{d^*(d^*+1)}\right)^s \left(\alpha^{d^*(d^*+1)}\right)^{\frac{n}{d^*+1} - s}. \tag{11}$$

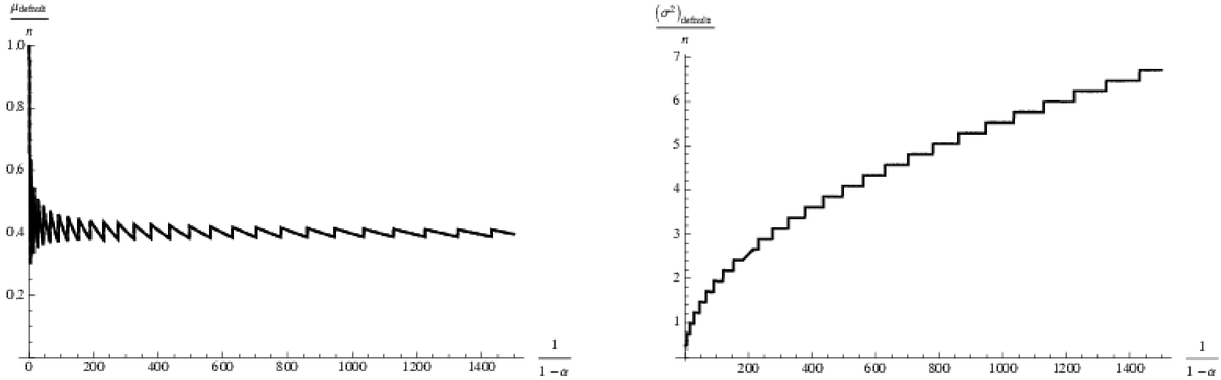
There is no first order stochastic dominance order among these distributions indexed by $\alpha$. However, the distributions with larger $\alpha$'s second order stochastically dominate those with smaller $\alpha$'s. Approximately, $s\,(-2log[\alpha])$ firms default out of $n$, with probability $\mathbb{F}\left[s, \left\lfloor n\sqrt{-2log[\alpha]}\right\rfloor, 1 - \boldsymbol{e}^{-0.5}\right]$, where $\mathbb{F}$ is the binomial pdf.

The mean and variance of the number of defaults are given by

$$\frac{\mu_{defaults}}{n} = \left(1 - \alpha^{d^*(d^*+1)}\right) \approx \left(1 - \boldsymbol{e}^{-0.5}\right),$$

$$\frac{\sigma^2_{defaults}}{n} = (d^* + 1)\left(1 - \alpha^{d^*(d^*+1)}\right)\left(\alpha^{d^*(d^*+1)}\right) \approx (-2log[\alpha])^{-0.5}\left(1 - \boldsymbol{e}^{-0.5}\right)\boldsymbol{e}^{-0.5}.$$

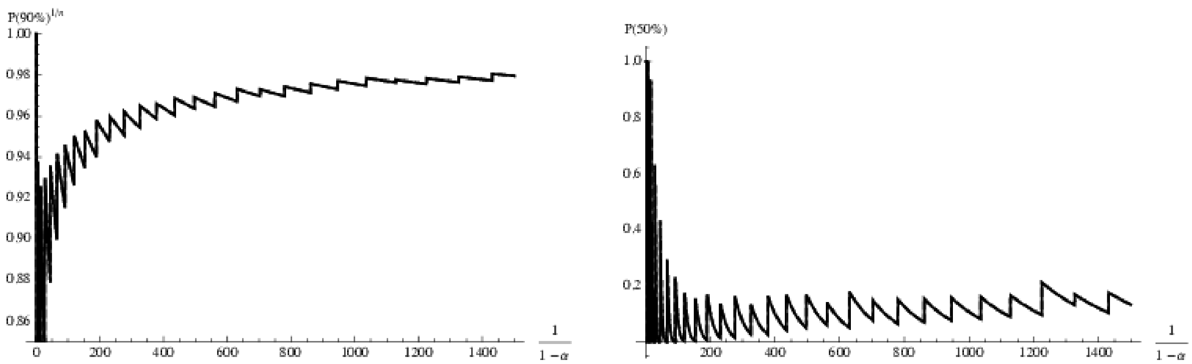
Except for the discreteness, the mean is constant but the variance gets larger in $\alpha$. It is worth noting that this is not a high mean and high variance case, but rather fixed mean, high variance situation. Increasing variance is due to the fact that firms in components have correlated risk and the components are getting larger in $\alpha$. Figure 13 show how the mean and variance vary with $\alpha$.



Mean Variance

Figure 13: Mean and variance of the number of defaults

There are other relevant notions of systemic risk besides the probability that all firms default. Figure 14 shows the probability that at least 90% of firms default and the probability that at least 50% of firms default. These notions of systemic risk also increase in $\alpha$, meaning that our volatility paradox insight is not particular to the notion of systemic risk defined as the probability of all firms defaulting.




90%, powered to $1/n$ %50

Figure 14: Probability that at least a certain fraction of firms default

Despite the constant mean and increasing volatility in the number of defaults, mean welfare increases in $\alpha$. The sum off payoffs is $nd^*\alpha^{d^*(d^*+1)}$ which is clearly increasing in $\alpha$ by definition of $d^*$ being the maximizer of $V(d)$. The distribution of welfare can also be pinned down. Realized ex-post welfare is $(n-(d^*+1)k)\,d^*\theta$ with probability given in Equation (11). Hence welfare has mean and variance given by

$$\frac{\mu_{welfare}}{n} = d^*\alpha^{d^*(d^*+1)}\theta \approx \boldsymbol{e}^{-0.5}\left(-2log[\alpha]\right)^{-1}\theta,$$

$$\frac{\sigma^2_{welfare}}{n} = (d^*+1)(d^*)^2\theta^2\left(1-\alpha^{d^*(d^*+1)}\right)\left(\alpha^{d^*(d^*+1)}\right) \approx (-2log[\alpha])^{-1.5}\left(1-\boldsymbol{e}^{-0.5}\right)\boldsymbol{e}^{-0.5}\theta^2.$$

Both means and variance are increasing in $\alpha$. These are plotted in Figure 15. Volatility paradox insight holds for welfare as well: welfare has higher mean and higher variance in fundamentally safer economies.

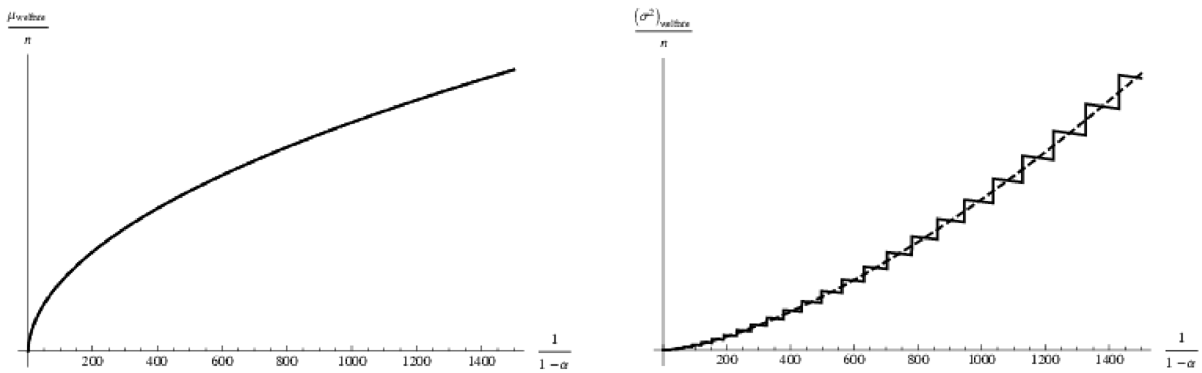



Figure 15: Welfare mean and variance

# Appendix C Stability under weak contagion

The subject network in Section 2.3.3 confirms that volatility paradox persists under weak contagion. There we relegated establishing the stability of the subject network to this Appendix. Recall that in Section 2.3.3 $V$ is given by Equation (4).

**Proposition 6.** *Suppose that* $\alpha^2 + \frac{\theta}{\beta} < 1$*. Then a network that consists of cliques of order* $d^*+1$ *is bilaterally stable.*

*Proof.* Let $\mathbb{F}$ and $\mathbb{G}$ denote the binomial PDF and CDF. Denote $\tilde{\theta} = \frac{\theta}{\beta}$. We have established earlier in Equation (4) that

$$V(d) = \beta\sum_{t=0}^{t=d}\mathbb{F}[t, d, 1-\alpha^d]\left(d\tilde{\theta}-t\right)^+$$

is the payoff of a firm in a clique with $d+1$ firms. If two firms in a clique of order $d^*+1$ cut a link, their payoff becomes

$$\beta\sum_{t=0}^{t=d-1}\mathbb{F}\left[t, d-1, 1-\alpha^d\right]\left(d\tilde{\theta}-\tilde{\theta}-t\right)^+ < V(d^*-1) < V(d^*).$$

This can not be profitable. If two firms in separate cliques of order $d+1$ add their missing link, their payoff becomes

$$W(d) = \beta \sum_{t=0}^{t=d} \mathbb{F}\left[t, d, 1-\alpha^d\right] \left( q(d) \left( d\tilde{\theta} + \tilde{\theta} - t \right)^+ + (1 - q(d) \left( d\tilde{\theta} + \tilde{\theta} - t - 1 \right)^+ \right)$$

where

$$q(d) = \begin{cases} \alpha^{d+1} \mathbb{G}\left[d\tilde{\theta}, d, 1-\alpha^d\right] & if\ (d+1)\tilde{\theta} < d \\ \alpha^{d+1} & if\ (d+1)\tilde{\theta} \geq d \end{cases}$$

We need to show that $V(d^*) \geq W(d^*)$.

For shorter notation and the ease of exposition, in the remainder of the proof, we use $\mathbb{G}_t$ for $\mathbb{G}\left[t, d^*, 1-\alpha^{d^*}\right]$, $\mathbb{F}_t$ for $\mathbb{F}\left[t, d^*, 1-\alpha^{d^*}\right]$, $q^*$ for $q(d^*)$, $m$ for $\lfloor d^*\tilde{\theta} \rfloor$, $s$ for $\lfloor d^*\tilde{\theta} + \tilde{\theta} \rfloor$.

*Case 1:* $m = s$.

$$\frac{W(d^*)}{\alpha V(d^*)} = \frac{\sum_{t=0}^{t=d^*} \mathbb{F}_t \left( q^* \left( d^*\tilde{\theta} + \tilde{\theta} - t \right)^+ + (1-q^*) \left( d^*\tilde{\theta} + \tilde{\theta} - t - 1 \right)^+ \right)}{\sum_{t=0}^{t=d^*} \mathbb{F}_t \left( d^*\tilde{\theta} - t \right)^+}$$

$$= \frac{\sum_{t=0}^{t=m-1} \mathbb{F}_t \left[ \left( d^*\tilde{\theta} + \tilde{\theta} - t \right) - (1-q^*) \right] + \mathbb{F}_m q^* (d^*\tilde{\theta} + \tilde{\theta} - m)}{\sum_{t=0}^{t=m} \mathbb{F}_t \left( d^*\tilde{\theta} - t \right)}$$

$$= \frac{\sum_{t=0}^{t=m} \mathbb{F}_t \left( d^*\tilde{\theta} - t \right) - \mathbb{F}_m (d^*\tilde{\theta} - m) + \mathbb{G}_{m-1}(\tilde{\theta} + q^* - 1) + \mathbb{F}_m q^* (d^*\tilde{\theta} + \tilde{\theta} - m)}{\sum_{t=0}^{t=m} \mathbb{F}_t \left( d^*\tilde{\theta} - t \right)}$$

$$= 1 + \frac{\mathbb{G}_{m-1}(\tilde{\theta} + q^* - 1) + \mathbb{F}_m \left( q^*\tilde{\theta} - (d^*\tilde{\theta} - m)(1-q^*) \right)}{\sum_{t=0}^{t=m} \mathbb{F}_t \left( d^*\tilde{\theta} - t \right)}.$$

If the numerator is negative, we are done. If it is positive:

$$\frac{W(d^*)}{\alpha V(d^*)} \leq 1 + \frac{\mathbb{G}_{m-1}(\tilde{\theta} + q^* - 1) + \mathbb{F}_m \left( q^*\tilde{\theta} - (d^*\tilde{\theta} - m)(1-q^*) \right)}{\mathbb{G}_{m-1}}$$

$$= 1 + \tilde{\theta} + q^* - 1 + \frac{\mathbb{F}_m \left( q^*\tilde{\theta} - (d^*\tilde{\theta} - m)(1-q^*) \right)}{\mathbb{G}_{m-1}}$$

$$\leq \tilde{\theta} + q^* + \frac{\mathbb{F}_m \left( q^*\tilde{\theta} - (d^*\tilde{\theta} - m)(1-q^*) \right)}{\mathbb{F}_{m-1}}$$

$$= \tilde{\theta} + q^* + \frac{d^* + 1 - m}{m} \times \frac{1 - \alpha^{d^*}}{\alpha^{d^*}} \times \left( q^*\tilde{\theta} - (d^*\tilde{\theta} - m)(1-q^*) \right).$$

Define $\epsilon = d^*\tilde{\theta} - m$.

$$\frac{W(d^*)}{\alpha V(d^*)} \leq \tilde{\theta} + q^* + \left(\frac{d^*+1}{d^*\tilde{\theta} - \epsilon} - 1\right) \times \frac{1-\alpha^{d^*}}{\alpha^{d^*}} \times \left(q^*\tilde{\theta} - \epsilon(1-q^*)\right).$$

Consider the function $\Phi$ of $\epsilon$ keeping all else fixed:

$$\Phi(\epsilon) = \left(\frac{d^*+1}{d^*\tilde{\theta} - \epsilon} - 1\right)\left(q^*\tilde{\theta} - \epsilon(1-q^*)\right).$$

$$\begin{aligned}\Phi'(\epsilon) &= (d^*+1)q^*\tilde{\theta}\frac{-1}{(d^*\tilde{\theta}-\epsilon)^2} - (d^*+1)(1-q^*)\frac{d^*\tilde{\theta}}{(d^*\tilde{\theta}-\epsilon)^2} + (1-q^*) \\ &< (d^*+1)q^*\tilde{\theta}\frac{-1}{(d^*\tilde{\theta}-\epsilon)^2} - (1-q^*) + (1-q^*) < 0.\end{aligned}$$

So $\Phi(\epsilon)$ is decreasing and maxed at $\epsilon = 0$. That is,

$$\frac{d^*+1-m}{m} \times \left(q^*\tilde{\theta} - (d^*\tilde{\theta} - m)(1-q^*)\right) < \frac{d^*+1}{d^*\tilde{\theta}} \times \left(q^*\tilde{\theta} - \epsilon(1-q^*)\right)$$

and

$$\frac{W(d^*)}{\alpha V(d^*)} \leq \tilde{\theta} + q^* + \left(\frac{d^*+1}{d^*\tilde{\theta}} - 1\right) \times \frac{1-\alpha^{d^*}}{\alpha^{d^*}} \times q^*\tilde{\theta} = \tilde{\theta} + q^* + \left(1 - \tilde{\theta} + \frac{1}{d^*}\right) \times \left(\frac{1-\alpha^{d^*}}{\alpha^{d^*}}\right) \times q^*.$$

*Case 1.1:* $d^*\tilde{\theta} \geq 1$. In this case, $1 - \tilde{\theta} + \frac{1}{d^*} \leq 1$ so that

$$\frac{W(d^*)}{\alpha V(d^*)} \leq \tilde{\theta} + q^* + \left(\frac{1}{\alpha^{d^*}} - 1\right) \times q^* = \tilde{\theta} + \frac{q^*}{\alpha^{d^*}} \leq \tilde{\theta} + \alpha \leq \frac{1}{\alpha}.$$

*Case 1.2:* $d^*\tilde{\theta} < 1$. Then $m = s = 0$. Since $s = 0$, $(d^*+1)\tilde{\theta} < 1$. Optimal $d^*$ can't be zero, so $(d^*+1)\tilde{\theta} < 1 \leq d^*$, hence $q(d^*) = \alpha^{d^*+1}\mathbb{G}_m = \alpha^{d^*+1}\mathbb{G}_0 = \alpha^{d^*+1}\mathbb{F}_0 = \alpha^{d^*+1+(d^*)^2}$. Then $W(d^*) = \beta\alpha^{d^*+1}\mathbb{F}_0 q^*(d^*\tilde{\theta} + \tilde{\theta}) = \beta\alpha^{2d^*+2+2(d^*)^2}(d^*\tilde{\theta} + \tilde{\theta})$. On the other hand $V(d^*+1) = \beta\alpha^{d^*+1+(d^*+1)^2}(d^*\tilde{\theta} + \tilde{\theta})$. Since $d^* \geq 1$, we have $2d^* + 2 + 2(d^*)^2 \geq d^* + 1 + (d^*+1)^2$ so that $V(d^*+1) \geq W(d^*)$, which implies $V(d^*) \geq W(d^*)$.

*Case 2:* $m = s - 1$.

*Case 2.1:* $(d^*+1)\tilde{\theta} < d^*$.

$$\frac{V(d^*)}{\beta\alpha^{d^*}} - \frac{W(d^*)}{\beta\alpha^{d^*+1}} = \sum_{t=0}^{t=d^*} \mathbb{F}_t \left(d^*\tilde{\theta} - t\right)^+ - \left[\sum_{t=0}^{t=d^*} \mathbb{F}_t \left(q^*\left(d^*\tilde{\theta} + \tilde{\theta} - t\right)^+ + (1-q^*)\left(d^*\tilde{\theta} + \tilde{\theta} - t - 1\right)^+\right)\right]$$

$$= \sum_{t=0}^{t=m} \mathbb{F}_t \left(d^*\tilde{\theta} - t\right) - \left[\sum_{t=0}^{t=s-1=m} \mathbb{F}_t \left(d^*\tilde{\theta} + \tilde{\theta} - t - 1 + q^*\right) + \mathbb{F}_s q^*(d^*\tilde{\theta} + \tilde{\theta} - s)\right]$$

$$= \mathbb{G}_m(1 - \tilde{\theta} - q^*) - \mathbb{F}_{m+1} q^*(d^*\tilde{\theta} + \tilde{\theta} - s).$$

$d^*\tilde{\theta} < m + 1 = s$ so $d^*\tilde{\theta} + \tilde{\theta} - s < \tilde{\theta}$. Also $(d^* + 1)\tilde{\theta} < d^*$, so that $q(d^*) = \alpha^{d^*+1}\mathbb{G}_m$. Also $\mathbb{F}_{m+1} < 1 - \mathbb{G}_m$. Plug all these in:

$$\frac{V(d^*)}{\beta\alpha^{d^*}} - \frac{W(d^*)}{\beta\alpha^{d^*+1}} > \mathbb{G}_m \left(1 - \tilde{\theta} - \alpha^{d^*+1}\mathbb{G}_m - (1 - \mathbb{G}_m)\alpha^{d^*+1}\tilde{\theta}\right).$$

$$> \mathbb{G}_m \left(1 - \tilde{\theta} - \alpha^{d^*+1}\mathbb{G}_m(1 - \tilde{\theta}) - \alpha^{d^*+1}\tilde{\theta}\right) > \mathbb{G}_m \left(1 - \tilde{\theta} - \alpha^{d^*+1}(1 - \tilde{\theta}) - \alpha^{d^*+1}\tilde{\theta}\right)$$

$$= \mathbb{G}_m \left(1 - \tilde{\theta} - \alpha^{d^*+1}\right) \geq 0.$$

That is $V(d^*) > \frac{1}{\alpha} W(d^*) > W(d^*)$.

*Case 2.2:* $(d^* + 1)\tilde{\theta} \geq d^*$. Then $s = d^*$, so that $m = d^* - 1$. Note that $(d^* + 1)\tilde{\theta} \geq \tilde{\theta}$ implies $(d^* + 1)\alpha^2 \leq 1$.

*Case 2.2.1:* If $d^* \geq 3$, then

$$W(d^*) < \beta\alpha^{d^*+1}(d^*\tilde{\theta} + \tilde{\theta}) \leq \beta\alpha^4(d^* + 1)\tilde{\theta} \leq \beta\alpha^2\tilde{\theta} = V(1) \leq V(d^*).$$

*Case 2.2.2:* If $d^* = 2$, then $s = 2$ so $\tilde{\theta} > \frac{2}{3} > \frac{1}{2}$. Then

$$\frac{V(2)}{\beta\alpha^2} = \alpha^4 2\tilde{\theta} + 2\alpha^2(1 - \alpha^2)(2\tilde{\theta} - 1)$$

$$= 2\tilde{\theta}\left[2\alpha^2 - \alpha^4\right] - 2\alpha^2\left(1 - \alpha^2\right) < 2\tilde{\theta}\left[2\alpha^2 - \alpha^4\right] - 2\alpha^2\tilde{\theta}$$

$$= 2\tilde{\theta}\left[\alpha^2 - \alpha^4\right] < \frac{\tilde{\theta}}{2} < \tilde{\theta} = \frac{V(1)}{\beta\alpha^2}$$

which is a contradiction. $d^* = 2$ is not possible in this case.

*Case 2.2.3:* If $d^* = 1$, then $s = 1$ and $\tilde{\theta} > \frac{1}{2}$.

$$\frac{W(1)}{\beta\alpha^2} = \alpha\left(\alpha^2 2\tilde{\theta} + (1 - \alpha^2)(2\tilde{\theta} - 1)\right) + (1 - \alpha)\left(\alpha^2(2\tilde{\theta} - 1)\right)$$

$$= 2\tilde{\theta}\left[\alpha + \alpha^2 - \alpha^3\right] - \left[\alpha + \alpha^2 - 2\alpha^3\right].$$

| Systemic risk | w.r.t.: source: | $\zeta$ (periphery node-shock) Central | Peripheral | Combined | $\zeta'$ (core node-shock) Central | Peripheral | Combined |
|---|---|---|---|---|---|---|---|
| If $d^* \leq m$ (small $\alpha$) | | Constant | Decreasing | Decreasing | Decreasing | Increasing | Decreasing |
| If $d^* > m$ (large $\alpha$) | | Constant | Inverse-U | Inverse-U | Decreasing | Constant | Decreasing |

Table 2: Central and peripheral systemic risk in node shock risk

$$\frac{V(1) - W(1)}{\beta\alpha^2} = 2\tilde{\theta}\left[\frac{1}{2} - \alpha - \alpha^2 + \alpha^3\right] + \left[\alpha + \alpha^2 - 2\alpha^3\right].$$

If the term in the first bracket is positive, we are done. If it is negative, then replace insert $\tilde{\theta} < 1 - \alpha^2$.

$$\frac{V(1) - W(1)}{\beta\alpha^2} > 2\left(1 - \alpha^2\right)\left[\frac{1}{2} - \alpha - \alpha^2 + \alpha^3\right] + \left[\alpha + \alpha^2 - 2\alpha^3\right]$$

$$= (1 - \alpha)\left(1 - \alpha^2\right) \geq 0.$$

□

# Appendix D Node shocks and systemic risk

As we have mentioned in Section 3.3.2, changing the probability of good node shocks, $\zeta$ and $\zeta'$, impacts the network as well. The comparative statics in $\zeta$ and $\zeta'$ are less interesting and the effects are summarized in Table 2. The only interesting case is when $\zeta$ increases for large core (large $\alpha$). Then, systemic risk has an inverse U-shape displayed in Figure 16. The reason that systemic risk initially increases is similar to the the peripheral volatility paradox introduced in Section 2.3.2. The reason systemic risk starts to decline after a threshold of $\zeta$ is that the risk of link shocks $1 - \alpha$ imposes an upper bound on the number links that peripheral firms want to form no matter how large $\zeta$ can be. Accordingly, when the periphery reaches this "satiation" point, the cliques do not expand any further and systemic risk stars falling as $\zeta$ increases.

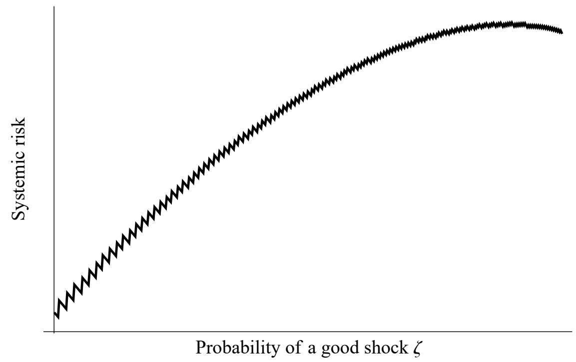



Figure 16: Volatility paradox with respect to the probability of a good node shock for the periphery